Invariants
Marin Math Circle

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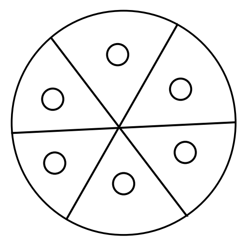
1. A circle is sliced into six slices by three diameters. A coin is placed in each of the slices. At each step two coins are chosen and moved to neighboring slices. Is it possible to collect all the coins in the same slice by such steps?



2. The numbers $1, 2, 3, 4, 5, 6$ are written on the board. In one step you can add 1 to any two numbers. Is it possible to get all the numbers to be the same by doing such steps?

3. Two friends are taking turns to break a chocolate bar of size 6×8. During each step it is allowed to break any of the pieces along a ridge. The persons who cannot make any more steps loses. Who will win in this game?

4. The numbers $1, 2, 3, \ldots, 19, 20$ are written on the board. At each step it is allowed to erase any two numbers a and b from the board and write $a + b - 1$ instead. What are the possible numbers that will be left on the board after 19 such operations?

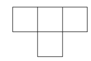
5. The numbers $1, 2, 3, \ldots, 19, 20$ are written on the board. At each step it is allowed to erase any two numbers a and b from the board and write $ab + a + b$ instead. What are the possible numbers that will be left on the board after 19 such operations?

6. Six pine trees are growing along a line. The distance between any two neighboring trees is 10 yards. There is one crow sitting on each tree. Crows can fly from tree to tree, but if a crow flies from a tree to another tree, another crow must fly the same distance but in the opposite direction. Is it possible for all six crows to gather on the same tree?

7. In the Bluegreywhite park there are 13 blue, 15 grey and 17 white chameleons. If any two chameleons of different colors meet, they both change their color to the third color. Is it possible for all the chameleons to have the same color at some point?

8. From an 8×8 chessboard two opposite corners are removed. Is it possible to tile the chessboard by dominos so that there is no overlap?

9. A "strong knight" moves on a 10×10 chessboard by a move of type $(1, 3)$, i.e. it can move by one cell in one direction, then turn 90 degrees and move another 3 steps (one more than a regular knight). Can a "strong knight" move to a neighboring cell by making several moves?

10. Is it possible to tile a 10×10 chessboard by tiles of the form  ? For which n is it possible to tile an $n \times n$ board by such tiles?

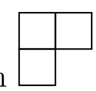
11. Is it possible to tile a 102×102 chessboard by 1×4 tiles?

12. Can a knight go through every cell of a $4 \times N$ board exactly once and return to where it started?

13. A dragon has 100 heads. A knight has two swords. With one he can cut 21 heads. With the other he can cut 3 heads, but if he does that, 2012 heads will grow back. Can the knight kill the dragon? The knight cannot use a sword if the dragon has less heads than the sword can cut.

14. The number 8^{2012} is written on the board. We erase it and replace it with the sum of its digits. If we repeat this procedure until a one digit number is left, what will that number be?

15. Given a triple of numbers the following operation is allowed: pick two, say a and b and replace them with $(a + b)/\sqrt{2}$ and $(a - b)/\sqrt{2}$. Is it possible to get to $(1, \sqrt{2}, 1 + \sqrt{2})$ from $(2, \sqrt{2}, 1/\sqrt{2})$ by repeatedly applying this operation?

16. For which n is it possible to tile the Aztec diamond of size n with tiles of the form  ?

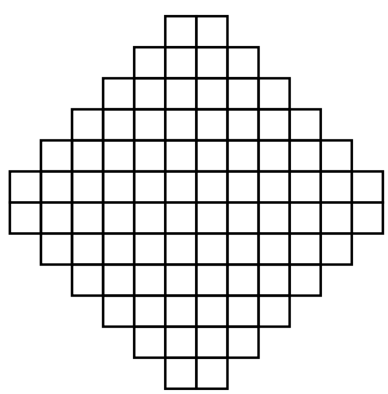


Figure 1: The Aztec diamond of size 6.

Note: Many problems are from Genkin, Itenberg, Fomin - "Leningrad math circles", 1994.