Counting and Symmetry

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This session owes much to Tom Davis, who wrote great notes on Polya's counting theory at

http://www.geometer.org/mathcircles/index.html

If you want to learn more after this session, the polya.pdf paper there is a great choice! Lots of my material here is shamelessly stolen (with permission) from that source.

My interest in this topic began with a problem from a well-known and widely-used precalculus textbook. I'll lead up to their problem (much harder than they thought it was!) with a few easier ones to get warmed up.

Problem 1. How many ways are there to put n different beads in order?

Problem 2. What if the beads are in a circle, so rotations don't matter?

Problem 3. What if the beads are on a bracelet, which can be rotated or flipped over?

Problem 4. Redo problem 1 through 3 with r red beads and g green beads. Hm, maybe that's too hard. Start with one red and one green, and work your way up. Ultimately the book's problem had 14 red and 6 green beads. Would it be much easier with 5 green instead of 6?

See how much harder it gets once there's symmetry?

Problem 5. How many ways are there to paint a six-sided die if you have one color of paint? Two colors? Three colors? n colors?

Problem 6. Now try the much harder problem: painting a cube with six identical faces.

Problem 7. How many different ways are there to put six identical beads around a bracelet? What if there are 5 of one color and 1 of another? Or 4, 2? Or 4, 1, 1? How about 3, 2, 1? How about all the ways using at most 3 different colors? (Analogy to chemistry)

Problem 8. My old favorite tie-dye shirt problem: how many ways are there to make a shirt with k vertical stripes if you have n colors to choose from? (First, what shirts are "the same"? Is this a hard problem even with no symmetry at all?)

Problem 9. Let's back up to some easier problems: how many ways are there to color a triangle with k colors? (Four possible answers: with no symmetry, rotation only, reflection only, or rotation and reflection.)

Problem 10. Same question, with a square.

Problem 11. How about an n-gon?

Problem 12. How many graphs are there on n vertices? (Start with small numbers, and compare doing it by hand with the fancier methods!)

There are several basic approaches here, that we used on all of the above problems. We didn't prove they are equivalent, but I hope you can see at least that it's plausible that they are.

First: count all the colorings, often n^k of them, where you have n colors to use in k spots. Then look at what colorings get overcounted: for instance, with two spots, you might call AB and BA identical, but AA has no such pairing. So all the pairs with two distinct colors need to get divided by two, and so on like that. This general technique—overcounting, and then dividing—works for all kinds of problems, not just counting with symmetry.

A second approach: A similar idea to the above, but more directly counting up each different symmetry class, instead of subtracting the excess.

A third approach: Polya counting. Look at the permutation group of your symmetries, and write the polynomial corresponding to it. For example, if you have a triangle with rotation but not reflection, you have $(a)(b)(c)$, (abc), and (acb) as the cycle decompositions of the three permutations. Then the polynomial is $(f_1f_1f_1 + f_3 + f_3)/3$, where the subscripts correspond to the number of elements in each cycle, and the dividing by 3 is because there are three permutations in total. Let $f_n = (x_1 + x_2 + \ldots + x_k)^n$. Then the coefficient of, say, $x_1x_1x_3$ is the number of ways of coloring the thing with two corners colored #1 and one corner colored #3; and hence the sum of the coefficients is the total number of ways of coloring the thing.

For small numbers of things, this machinery is definitely the hard way to do it, but for giving general solutions or for large and complicated counting, Polya is far more efficient than the other approaches. I hope the problems on the previous page were enough to convince you of that!