Putting Things in Order

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1 Warm-up Problems

Only one of these has anything to do with today's topic, but I like them all! We'll see how much time we have for them.

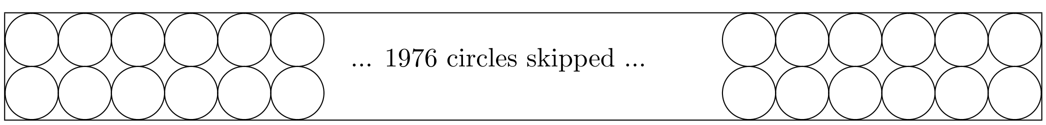
Problem 1 A group of one hundred students, with no two exactly the same height, were arranged in a square formation. In each of the ten rows, the shortest student raised his or her hand – of these students, John was the tallest. Then, in each of the ten column, the tallest student raised his or her hand; of these, Mary was the shortest. Who is taller, John or Mary?

Problem 2 A mathematician is lost in the woods. The woods are a long rectangular strip, 100 miles long and 1 mile wide. She does not know where she is in the woods or which direction she is facing. Find the shortest path you can that is guaranteed to bring her out of the woods. Your answer should include the maximum length shell have to walk, a description of the pattern by which she moves, and some evidence that it is guaranteed to bring him to the edge of the woods. For example:

> If she walks 1 mile in any direction, then turns 90 degrees left and walks another mile, then turns 90 degrees left and walks another mile, then turns 90 degrees left and walks another mile, she will have traversed a square of side length 1 mile. Since the strip is only 1 mile wide, this square cant be totally inside the strip, so at some point, she must reach an edge of the woods, walking at most 4 miles to do so.

Can you do better?

Problem 3 Show how to fit the maximum number (or at least as many as you can) of non-overlapping circles of diameter 1 inside a rectangle of dimensions 2×1000.



It isn't hard to fit 2000 such circles in the rectangle in two rows of 1000 circles each as illustrated above. Can you do better?

2 Introduction to tonight's main topic

These notes are fragmentary; I haven't written every proof and definition in detail – I'd like to work the details out together with you in class!

Linear ordering may seem like a small topic – after all, it's just a bunch of objects, placed in a row! But they can be surprisingly subtle, and they crop up in a lot of branches of mathematics – among them, set theory, logic, recursion theory, topology, analysis, graph theory and combinatorics.

Today the *main* goal is to learn enough to appreciate and prove a theorem about partial orders and then see how it can be applied to a number of seemingly-unrelated theorems in graph theory, combinatorics, and linear algebra.

3 Total Order and order types

The *formal* definition of a linear order looks like:

Definition 1 *A* linear ordering of the set *A is a binary relation $\preceq$ on A satisfying the conditions for any a, b, and c in A:*

transitivity *if $a \preceq b$ and $b \preceq c$, then $a \prec c$.*

reflexivity *$a \preceq a$.*

antisymmetry *if $a \preceq b$ and $b \preceq a$, then $a = b$.*

connectivity *For all a and b, either $a \preceq b$ or $b \preceq a$.*

Other terminology: A linear order is sometimes called a *total order*. Also a *chain* (see next section)

We should try to think of some different examples to make clear what this definition means. **Examples:** The integers, the real numbers, the rational numbers, ordered the way we always order them. But we can think of other examples, like all polynomials with real coefficients, or words in a dictionary – wait, can we compare them to real numbers? (Lexicographic order on sequences) What if the words were infinitely long?

Are two total orders of the same size *isomorphic*? What does "isomorphic" mean?

Notice that connectivity implies reflexivity. Why did we bother to include reflexivity as an axiom?

Problem 4 Show any countable linear order is *embeddable* (what does "embeddable" mean?) in the rationals, under the usual order.

Problem 5 If $\langle A, \preceq_1 \rangle$ is embeddable in $\langle B, \preceq_2 \rangle$ and $\langle B, \preceq_2 \rangle$ is embeddable in $\langle A, \preceq_1 \rangle$, does that mean that the two must be *isomorphic* (what does that word mean?) ?

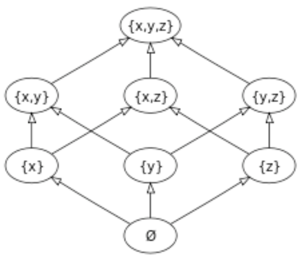
4 Partial Orders

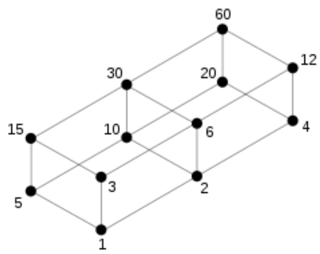
Definition 2 *A* partial order *of the set A is a binary relation relation $\preceq$ on A that satisfies transitivity reflexivity, and antisymmetry, but not necessarily connectivity.*

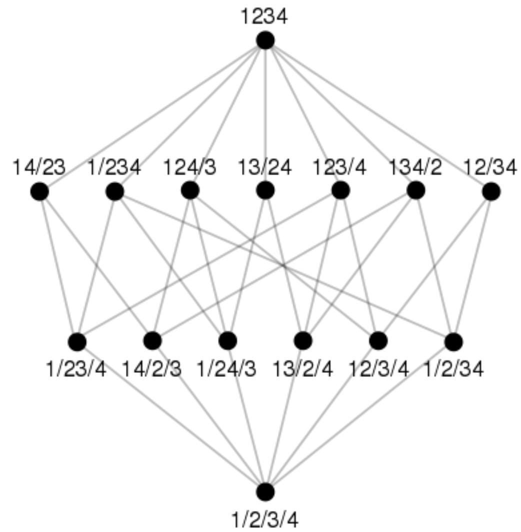
Other terminology:

- some people call a partially ordered set a *poset*.
- a *weak partial order* satisfies transitivity and reflexivity, but not necessarily antisymmetry.
- Elements a and b of the set A are said to be *comparable* if either $a \preceq b$ or $b \preceq a$. (so, in a linear order, all pairs of elements are comparable). Elements that are not comparable are called *incomparable*.
- a *chain* is a subset of partially ordered set in which the partial order is linear [that is, all pairs of elements of the subset are comparable].
- An *anti-chain* is a subset of the partially ordered set in which no two distinct elements are comparable (The elements are pairwise incomparable).

Rather than formally define them, I'm just going to draw some *Hasse diagrams*.

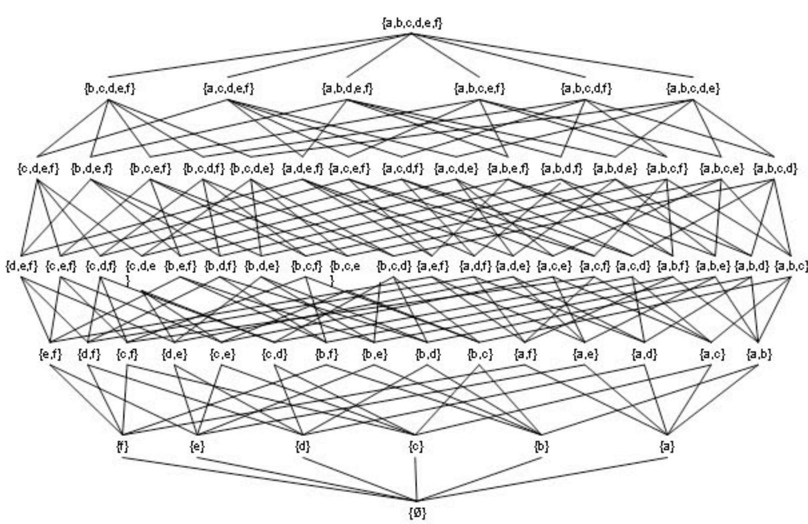






Examples.

- Set inclusion (this is a good one to think about!). Can apply this to subgroups of a group, or other contexts.
- integers and divisibility
- a sequence of tasks that must be done (by construction workers, by a computer programs running in parallel, by an airline scheduling flights), ordered by which must be done before others are done.
- SAT scores, or more generally n-tuples. Various ways to to define $\vec{x} \preceq \vec{y}$ to induce a partial order.
- real-valued functions, with various ways to define $f \preceq g$ to induce a partial order.
- a directed (acyclic) graph (what's that?) ordered by reachability (what's that?).
- the students in the problem below, with an ordering based on their height *and* their position in the line
- (good example, but might take more machinery to explain than we have time for today) Partitions of a set, ordered by coarseness. [This can be viewed in terms of set inclusion by looking at a partition of a set as a set of subsets of the original set, closed under union and intersection].



Question (not covered here!): how many different partial orders on n elements are there? (For total orders this is a much simpler question!)

Question: (not hard). Can every partial order be extended to a total order? What does "extended" mean here?

Question (not covered here!): Given a partial order, how many ways can it be extended to a total order?

Open question (1/3-2/3 Conjecture): In any partially ordered set that is not a linear order, there is some pair (x, y) of elements such that the proportion of linear extensions in which x is above y lies between 1/3 and 2/3.

Problem 6 The same hundred students mentioned in the warmups above, still with no two of exactly the same height, were marching in a single column. Prove that *either* you can find ten students (not necessarily consecutive) in the column whose are in ascending order (that is, the smallest student is in front of the others, the second smallest is next, etc.) *or* you can find *twelve* students (again, not necessarily consecutive) in the column whose heights are in *descending* order (that is, the tallest is in front, etc.).

This result (well, the slightly more general statement of it) is known as the ErdősSzekeres theorem.

Problem 7 Given an infinite sequence of rational numbers, show there is either a monotonically increasing (infinite) subsequence, a monotonicaly decreasing (infinite) sequence, or a constant (infinite) sequence.

5 Dilworth's theorem

5.1 Statement of the theorem

Theorem 3 (Dilworth) *If every antichain in a (finite) partially ordered set has at most m elements, then the set may be partitioned into m chains.*

Actually, the set doesn't need to be finite, though m does. The proof given today will just be for finite sets, though.

The converse is pretty immediate. If a set may be partitioned into m chains, no antichain can have more than m elements.

Problem 8 [(All [Soviet] Union Mathematical Olympiad 1972)] Fifty line segments lie on a common line. Show that either some eight of the segments have a non-empty intersection, or eight of the segments are pairwise disjoint. (We can do this with Dilworth's theorem! How do you suppose we might put a partial ordering on line segments?)

5.2 Before the proof of the theorem

First, there's a theorem that seems similar but is much easier to prove:

Theorem 4 *If every chain in a partially ordered set has at most m elements, then the set may be partitioned into m antichains.*

This can be proved by defining a concept: the *height* of an element. How do you think this should be defined? And how does that lead to a proof of this theorem? (And why doesn't that proof carry over to Dilworth's theorem?)

5.3 Proof of Dilworth's theorem

Some ways to prove Dilworth's theorem: Could use induction on edges of the associated graph, we'll use induction on the size of the partially ordered set. Should we have a brief digression on induction?

Base case: With only one element in the partially ordered set, Dilworth's theorem is pretty simple! (For that matter, if no antichain contains more than one element, things are still pretty easy no matter how many elements are in the set).

Induction step. Suppose we've managed to show that Dilworth's theorem is true for all partially ordered sets with at most n elements. Now let's take any partially ordered set S with $n+1$ elements, for which every antichain in the set has at most m elements. We're going to find a nice way to remove some elements from the set so that we can apply Dilworth's theorem to the smaller set. [In this version of the proof, we'll actually apply Dilworth's theorem to *two* smaller sets that can be glued back together again in a simple way. There are many other ways to prove the theorem.]

Start by finding a maximal chain C in our set. (Wait, what is that?) We then look at the *rest* of our set $S \setminus C$, and apply Dilworth's theorem to it. It certainly has no antichains of size *larger* than m (why?). If the largest antichain in $S \setminus C$ is of size $m-1$ or smaller, we're home free (why?). So the only tricky case is if there is an antichain $A = \{a_1, a_2, \ldots, a_m\}$ in $S \setminus C$.

Claims to verify:

- A is a maximal antichain in S. In particular, every element in C is comparable to at least on element in this antichain.
- if we define $U = \{x \,|\, a_i \preceq x$ for at least one a_i in the antichain $A\}$ and $L = \{x \,|\, x \preceq a_i$ for at least one a_i in the antichain $A\}$, then $U \cap L = A$, and at least one element of C is in U and at least one element of C is in L

Then $A \cup U$ has size less than $n+1$, and a largest antichain of size m, so we may apply the induction hypothesis to it, resulting in partition of $C \cup U$ into m chains. Similarly, we can do the same with $A \cup L$.

Finally note that the elements of A must be the minimal elements of the chains in U, and they must be the maximal elements of the chains in L, so we can 'glue together' the chains to get a partition of $L \cup U$, which is all of S, into m chains.

5.4 Other graph theory theorems that can be reduced to Dilworth's theorem

Dilworth's theorem is closely related to many theorems in graph theory and even combinatorics problems that don't sound like they have anything to do with partial orderings:

For example: Dilworth's theorem gives a pretty quick proof to Hall's Marriage Theorem, too:

Theorem 5 (Hall's marriage theorem) *Given a collection of people with n men and n women with the property that, for any subset of k men (where k could be any integer between 1 and n), there are at least k women known to at least one man in the subset, then there must be a way to pair each man with a distinct woman known to him.*

Sketch of proof: Define a partial ordering in which each man is "less than" every woman he knows. How big are the biggest antichains?

Problem 9 In a game of solitaire, an ordinary deck of 52 playing cards is dealt into 13 piles with four cards each. Is it possible to re-order the cards in each pile such that the collection of top cards in the 13 piles represent one card from each rank (ace, 2, 3, ..., king), as does the collection of second cards from each pile, and the third cards, and the fourth? Or, if a similar game is played with four piles, each with 13 cards, can one re-order the cards so that the top cards from four piles contain one from each suit, as does the collection of second cards, etc.?

Theorem 6 (Kőnig's theorem) *In a bipartite graph, the number of edges in a maximum matching equals the number of vertexes in a minimum vertex cover.*

What's a graph? A *bipartite* graph? a maximum matching? a minimum vertex cover?

This is sort of a generalization of Hall's marriage theorem, but it's possible to prove it using Dilworth's theorem, too.

Theorem 7 (Birkhoff-von Neumann) *Any doubly-stochastic matrix may be represented as a convex combination of permutation matrices*

Wait! What's a doubly-stochastic matrix? What's a permutation matrix? What's a convex combination? And what on earth does this have to do with partial orders? [actually, I'm not sure that it does have anything to do with them, but I do know a proof that uses the Hall Marriage Theorem. But what on earth does this have to do with men and women? A permutation matrix "marries" each row to a column.]

While we're at it: How big is the biggest anti-chain for the partial order defined by inclusion on the set of all subsets of n elements?

Theorem 8 (Sperner) *if $\mathcal{A}$ is a family of subsets of n elements with the property that no element of the family is contained in any other, then*

$$|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$$

(and this is achieved by letting $\mathcal{A}$ be the collection of all subsets of size $\lfloor n/2 \rfloor$).

Proof, via the LYM (Lubell, Yamamoto, Meshalkin) inequality:

$$\sum_{A \in \mathcal{A}} 1/\binom{n}{|A|} \leq 1$$

Theorem 9 (Elias-Feinstein,Shannon, also Ford-Fulkerson) *In a flow network with source s and sink t, the maximum achievable flow is equal to the minimum capacity of a cut.*

What's a flow network? a source? a sink? a flow? a cut? a capacity? Is this result trivial? What does it have to do with partial orders? (It helps to only use integers in the capacities, but this is not a serious problem.)

6 Well-orders

(Don't expect to get to this section)

Definition 10 *A* well-ordering *of a set A is a total ordering of A in which every subset of A has a minimal element.*

Examples:

We can (and may) also talk about *well-partial-orderings*, whose definition is very similar to the above.

In a well-ordering, every element has a *successor*, an element that follows it in the ordering without any others in between.

Is that sufficient to be a well-ordering?

Problem 10 If well-ordering $\langle A, \preceq_1 \rangle$ is embeddable in $\langle B, \preceq_2 \rangle$ and $\langle B, \preceq_2 \rangle$ is embeddable in $\langle A, \preceq_1 \rangle$, does that mean that the two must be isomorphic?

Problem 11 If f is an order-preserving function from a well-ordered set to itself, show that $x \preceq f(x)$ for all x in the set.

Problem 12 (harder) If $\langle A, \preceq \rangle$ is a total order with the property that every order preserving function f from the set to itself satisfies $x \preceq f(x)$ for all x in the set, must A be well-ordered?