Marin Math Circle π-Day, 2012

HOW TO GAMBLE IF YOU MUST

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Fun with Dice

1 *Thirty-six scenarios*. When two dice are rolled, there are 36 different outcomes, because $36 = 6 \times 6$. Each outcome is equally likely. This allows us to compute probabilities easily. Use this table to compute the various sums that can occur. I filled in the first few cells.

	1	2	3	4	5	6
1	2	3	4			
2		4				
3						
4						
5						
6						12

(a) Verify that the probability of rolling two dice and getting a sum of 2 is $1/36$.

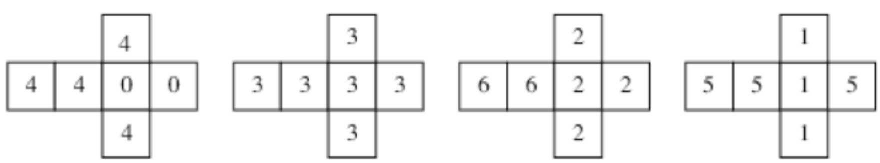
(b) What is the probability of rolling two dice and getting a sum of 5?

(c) What is the most likely sum, and what is its probability?

(d) Test these probabilities by experiment. For example, suppose you want to test for a sum of 12, which should also have a probability of $1/36$. You'd expect to see this sum happen only once every 36 or so rolls. Experiment!

2 *Non-transitive dice*. Construct the following set of four dice.

A B C D



(a) Suppose one person tosses the A die, and the other tosses the B die. The winner is the one whose die has the bigger number. What is the probability that A wins? You will need to make a 6×6 table like this. I filled in a few cells, saying who the winner is.

A vs B	0	0	4	4	4	4
3	B	B	A			
3						
3						
3						
3						
3						

(b) Work out the probabilities for the other possibilities, such as B vs. C, C vs. D, etc. Here are some tables to help you organize your work.

B vs. C	2	2	2	2	6	6
3	B	B			C	
3						
3						
3						
3						
3						

C vs. D	2	2	2	2	6	6
1	C	C				
1						
1						
5						
5	D					
5						

A vs. C	2	2	2	2	6	6
0						
0						
4						
4						
4						
4						

A vs. D	1	1	1	5	5	5
0						
0						
4						
4						
4						
4						

(c) Now that you have collected data, do you notice something strange about these four dice? Can you construct a "sucker bet" with them?

3 What's the probability of rolling six dice and

(a) getting a sum of 6?

(b) getting a sum of 7?

(c) getting a sum of 10?

(d) having all six numbers be equal?

(e) having all six numbers be different?

4 *How to resolve a Love Triangle*. Akira, Betsy and Cleopatra fight a 3-cornered pistol duel. All three know that Akira's chance of hitting any target is $1/3$, while Betsy *never* misses, and Cleopatra has a 0.5 chance of hitting any target. The way the duel works is that each person is to fire at their choice of target, starting with Akira, and proceeding in alphabetical order (unless someone is hit, in which case they don't shoot), continuing until one person is left unhit. What is Akira's strategy?

Experiment with different strategies, and simulate the shooting using your dice. You can simulate an event with probability $1/3$ by tossing one die and seeing if the the number shown is 1 or 2, say. Likewise, you can come up with ways to simulate a $1/2$ probability event (you don't need a coin, you can still use a die).

Here are some possible strategies for Akira: shoot at Cleopatra first; shoot at Betsy first; try something else. Choose a strategy, and try simulating the duel. See if you can experimentally estimate the probability that Akira survives, using various strategies.

5 (a) On average, how many times must a die be thrown until one gets a 6?

(b) How many times, on average, should one toss a fair die in order to see all 6 possible outcomes?

Fun with Cards

1 I deal cards onto a table, while you tell me whether the card should be face up or face down (I do what you say). You also tell me when to stop dealing the cards (it doesn't have to be the whole deck). Then you put a blindfold on me. I bet you that my fingers are so sensitive that I can detect the difference between a face up card and a face down card by touch. I bet that I can divide the cards on the table into two piles, each of which has exactly the same number of face cards. Do you want to take this bet?

2 A deck of cards is randomly cut into three piles. I bet that at least one of the cards on the top of a pile is a "face card;" i.e., a Jack, Queen, or King? Do you want to take this bet?

3 We each have a shuffled deck of cards and we deal our cards one at a time and compare. I bet you that we will have at least one match (for example, our 7th cards are the same). Do you want to take this bet?

4 *Kruskal Count*. A deck of cards is shuffled. You start dealing the cards. Each card gets a numerical value: face cards are 1, number cards 1–5 are their value, and number cards 6–10 are five less. For example, a Jack equals 1 and a 7 equals 2.

You deal the first card, and if it equals n, you then deal out n more cards. You look at this last-dealt card, and if it equals m, you deal out m more cards. For example, suppose the first card is a 3. Then you deal out 3 more, and suppose that the last card (the 4th) is a 9. Then you deal out 4 more, etc. Do this 7 times. In other words, you have looked at a card, noticed its value, then dealt out that many cards, and you did this 7 times.

Now, keeping the cards in order, have someone do the exact same procedure, except that they start with the *second* card in the deck, instead of the first card. Again, read the card and deal that many more, read the last card, etc., and do this 7 times.

Do you think that you will end up at with the same card as when you started with card #1? Is it a good bet to make?

Problems Involving Expectation, Random Walks, and Others

1 *The Classic Birthday Problem.* In a room of 30 people, how likely is it that at least two people have the same birthday?

2 *Another Birthday Problem.* In Klopstockia, all factory workers must work every day of the year, getting no days off, with only one exception: whenever it is a worker's birthday, the factory closes down and everyone gets the day off! Factories are obligated to hire workers randomly, without discriminating on the basis of birthday. In order to maximize output, how many workers should a factory employ?

3 It costs a consumer $1 to buy a Klopstockia lottery ticket. The buyer then scratches the ticket to see the prize. Compute, to the nearest penny, the expected profit that the state of Klopstockia makes per ticket sold, given the following scenarios for prizes awarded. (The state will make a profit if the expected value of the lottery ticket is *less* than $1.)

(a)

Prize	$1	$10	$1000
Probability	$\frac{1}{10}$	$\frac{1}{1{,}000}$	$\frac{1}{1{,}000{,}000}$

(b)

Prize	$1	$10	$1000	a free lottery ticket
Probability	$\frac{1}{10}$	$\frac{1}{1{,}000}$	$\frac{1}{1{,}000{,}000}$	$\frac{1}{5}$

4 Two players alternately toss a penny, and the one that first tosses heads wins. What is the probability that

(a) the game never ends?

(b) the first player wins?

(c) the second player wins?

5 *Hint: Expectation is Additive*.

(a) Place n letters at random into n envelopes. What is the average number of letters which get into the correct envelopes?

(b) Shuffle an ordinary deck of 52 playing cards. On the average, how far from the top will the first ace be?

(c) An urn contains 1000 balls, labeled $1, 2, 3, \ldots, 1000$. You perform the following procedure: mix the balls well, pick a ball, record its number, and then put it back in the urn. If you do this procedure 1000 times, what is the average number of *distinct* integers that you will record?

6 *The Classic Gambler's Ruin Problem.* Two players take turns tossing a fair coin. If the coin is heads, player A gives player B a dollar. If the coin turns up tails, B gives A a dollar. Player A starts with a dollars, and player B starts with b dollars (a and b are non-negative integers). Once a player goes bankrupt (i.e., has zero dollars) the game is over. What is the probability that A goes bankrupt?

What happens if the probabilities are not equal; i.e., what if the probability that the coin is heads is p, for some fixed $0 \le p \le 1$.

7 *A Bug on a Cube.* Imagine a bug that crawls along the edges of a cube. The bug does not change directions while traveling on an edge. Two adjacent vertices, F and P, have food and poison, respectively. If the bug reaches either of these vertices, it stops traveling. Whenever the bug reaches one of the other six vertices, it has a choice of three edges on which to travel and it chooses randomly (i.e., with probability of $1/3$ for each choice). For each of these six starting vertices, compute the probability that the bug lives (i.e., reaches F before reaching P).

8 *What a Loser!* You arrive in Las Vegas with \$100 and decide to play roulette, making the same bet each time, until you are either bankrupt or have doubled your money. Which of the following strategies is best?

(a) Making bets of \$1 each time.

(b) Making bets of \$10 each time.

(c) Making a single bet of \$100.

9 *A Gambling "System."* Suppose you are playing a game with a 50% chance of winning each time. You can bet any amount, and if you win, you win twice your bet. If you lose, you lose your bet. In other words, if you bet B dollars, your *profit* is $\pm B$ depending on whether you win or lose. You decide that you will play, stopping as soon as you win, doubling the size of your bet each time. You are guaranteed to make a profit? Right? Use expectation to show that this won't work. What if you triple instead of double?

10 *The St. Petersburg Paradox.* Consider the following game. I will flip a fair coin until it shows up heads. We keep track of the number of flips until this happens. If it happens on the first flip, I'll pay you \$2. If it takes two flips, then I'll pay you \$4. Three flips, \$8, etc. In other words, if it takes n flips until the first head, I will pay you 2^n dollars. Pretty sweet game!

How much is this game worth *to you*? In other words, if a there was a ticket that allowed you to play the game once with me (I flip the coin until it is heads, and pay you the appropriate amount), how much would you pay for the ticket? Clearly, you'd pay at least 1 dollar. In fact, you'd almost certainly pay 2 dollars. How about 3? 4? 5? More?

11 *A Problem from The 2000 Bay Area Math Meet (BAMM).* Consider the following experiment:

- First a random number p between 0 and 1 is chosen by spinning an arrow around a dial which is marked from 0 to 1. (This way, the random number is "uniformly distributed"—the chance that p lies in the interval, say, from 0.45 to 0.46 is exactly $1/100$; and the chance that p lies in the interval from 0.324 to 0.335 is exactly $11/1000$, etc.)
- Then an unfair coin is built so that it lands "heads up" with probability p.
- This coin is then flipped 2000 times, and the number of heads seen is recorded.

What is the probability that exactly 1000 heads were recorded?