Angle Defect, Curvature, and the Gauss-Bonnet Theorem

Marin Math Circle
Linda Green
linda@marinmathcircle.org

March 21, 2012

Parts of this handout are taken from *Geometry and the Imagination* by John Conway, Peter Doyle, Jane Gilman, and Bill Thurston. See http://www.geom.uiuc.edu/docs/doyle/mpls/handouts/handouts.html

1 Triangulations

1. What surface do you get if you glue (tape) together equilateral triangles putting three triangles around every vertex? Four triangles around each vertex? Five triangles around each vertex? Six triangles around each vertex? Seven triangles around each vertex?

2 Angle Defect

The angle defect at a vertex of a polygon is defined to be 360° minus the sum of the angles at the corners of the faces at that vertex. For instance, at any vertex of a cube there are three angles of 90°, so the angle defect is $360° - 270° = 90°$. You can visualize the angle defect by cutting along an edge at that vertex, and then flattening out a neighborhood of the vertex into the plane. A gap will form where the slit is: the angle by which it opens up is the angle defect.

The total angle defect of the polyhedron is the sum of the angle defects at all the vertices of the polyhedron. For a cube, the total angle defect is $8 \times 90° = 720°$.

2. What is the sum of the interior angles of a polygon (in the plane) with n sides?

3. Determine the total angle defect for each of the 5 regular polyhedra and for a triangular prism.

(Semi) regular polyhedron	Angle defect of each vertex	Number of vertices	Total angle defect
Triangular prism			
Tetrahedron			
Octahedron			
Cube			
Icosohedron			
Dodecahedron			
Soccer ball			

4. Build (or imagine) a polyhedron in the shape of a torus (a donut). Calculate its total angle defect.

3 Descartes Angle Defect Formula

5. The total angle defect is intimately connected with another number from topology: the Euler number. What is the relationship and why does it hold?

6. Why is the Euler number the same for all polyhedra that form the same topological surface?

4 Curvature

If you take a piece of the skin of a sphere, you can't flatten it onto a plane without either stretching it or tearing it. Try it with an orange peel. Even a small piece needs to be ripped to flatten on the table. This is because a piece of orange peel has a different geometry from a piece of flat paper. The piece of a orange peel is curved, and the paper is not.

7. Which seems more curved, a piece of the surface of an orange or a piece of the surface of a watermelon? The whole surface of the orange or the whole surface of the watermelon?

   One way to measure the curvature of a piece of a surface is to cut a narrow ring from the boundary, cut the ring open into a strip and flatten this strip onto the table, so that it opens up. The total (Gaussian) curvature of the piece of surface is the angle by which the strip opens up. This angle is closely related to the angle defect of a polyhedron.

   If the strip meets up with itself perfectly, then the region has zero curvature. Sometimes, the strip doesn't meet up because it doesn't curl enough. This is positive curvature. Sometimes the strip doesn't meet up because it curls around too much and overshoots. This is negative curvature.

8. What is the curvature of a region of a flat piece of paper?

9. What is the curvature of a piece of the surface of a cylinder?

10. Measure the curvature of some of these vegetables and fruits:

    - cabbage
    - kale
    - lettuce
    - mustard greens
    - orange peel
    - banana peel
    - potato peel

    You'll need to pay attention not only to the angle, but also to how the strip curls around, keeping in mind that zero curvature is a strip that comes around and meets itself. Be careful about 180°'s and 360°'s. See the figure on the next page.

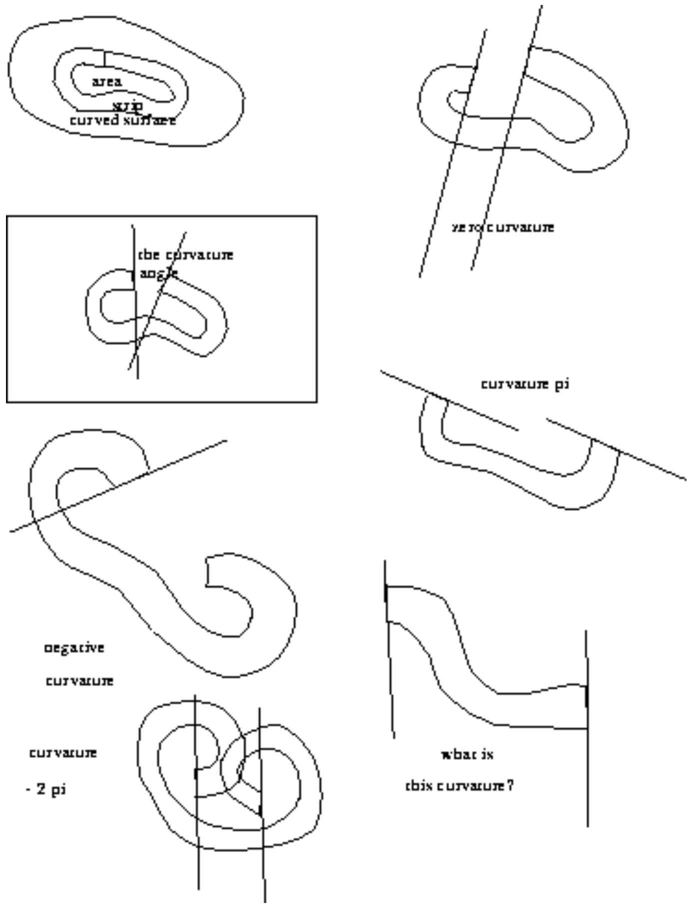

Figure from *Geometry and the Imagination* by John Conway, Peter Doyle, Jane Gilman, and Bill Thurston. See http://www.geom.uiuc.edu/docs/doyle/mpls/handouts/handouts.html

11. If you take two adjacent pieces of a surface, is the total curvature in both pieces put together the same as the sum of the curvature in each piece? Why?

12. What is the curvature of a piece of a sphere on the *outside* of a tiny circle?

13. On a polyhedron, what is the curvature inside a region containing one vertex? Two vertices? All but one vertex? All the vertices?

14. Construct a surface from equilateral triangles by putting seven triangles around each vertex. What is the curvature of a piece of this surface containing one vertex? Three vertices?

5 The Celestial Sphere

Imagine walking around on a surface with a flashlight pointed straight overhead at all times. The flashlight sweeps out a pattern on the "celestial sphere" which you can imagine as a large sphere surrounding you and the surface.

15. What pattern is swept out on the celestial sphere if you travel on the surface of a cube, walking in a loop around each vertex in turn? What if you travel around each vertex of an icosohedron?

16. On a convex polyhedron in which three faces meet at each face, the celestial image of a path around a vertex is a triangle. Show that the three angles of this celestial triangle are the supplements of the angles of the faces meeting at the vertex.

17. Show that the area of this celestial triangle is the angle defect at the vertex. You will need to measure angle defect in radians instead of degrees, and you will need to use Girard's formula: area of a

triangle on a sphere = sum of angles - π. Assume that the radius of the celestial sphere is 1 unit in some weird measurement system.

The curvature of a piece of a surface – any surface, not just a polyhedron – can also be defined as the area inside the celestial image of the piece of the surface.

6 The Gauss Bonnet Theorem

The Gauss Bonnet Theorem generalizes Descartes Angle Defect Formula to surfaces that are not polyhdera. It says that the total curvature of any closed surface S is $2\pi\chi(S)$, where $\chi(S)$ is the Euler number of S and 2π is the same thing as $360°$, just written in radians.

The Gauss Bonnet Theorem is amazing because it relates curvature (geometry) to Euler number (which depends only on topology). It tells us that even if God or earthquakes build new mountains on earth, creating additional positive curvature, that new positive curvature has to be exactly balanced by new saddle points, with negative curvature, so that the total curvature remains unchanged.