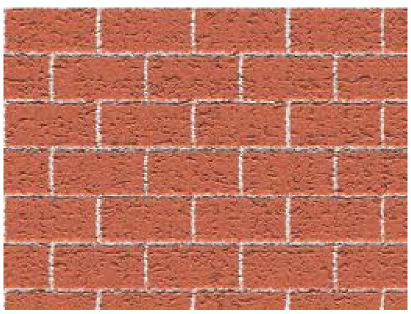
Tilings **Marin Math Circle**

linda@marinmathcircle.org *Sept. 30, 2012*

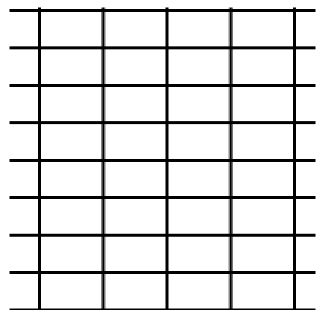
Most of this material is from James Tanton's Newsletter: Mathematical Musings from the St. Mark's Institute of Mathematics, September, 2010.

Definition A polygon **tessellates** the plane if it is possible to cover the entire surface of the plane with congruent copies of the polygon, with no gaps or overlaps (except along the edges of the polygons). The tessellation is called a **tiling** if each edge of the polygon matches an entire edge of the adjacent polygon.

We're all familiar with some of the ways that rectangular bricks can tesselate or tile the plane.



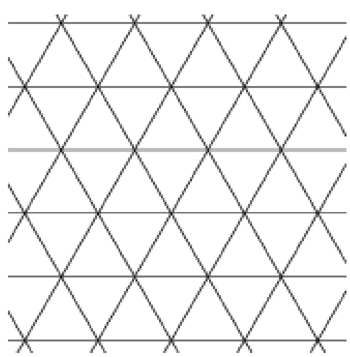
A tessellation.



A tiling.

1. What are some other polygons tile the plane?

   Definition A tiling is called **periodic** if it has translational symmetry in (at least) two non-parallel directions. That is, you can shift the whole tiling in some direction and it will line up exactly with itself, and you can do the same thing in some second, non-parallel direction.



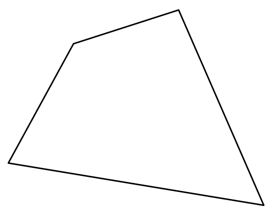
Periodic tiling with equilateral triangles.

2. Does every triangle tile the plane in a periodic way?

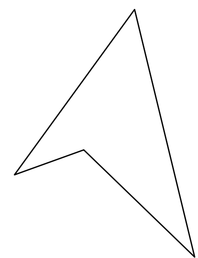
3. Is it possible to use a triangular tile to create a non-periodic tiling of the plane?

4. The definition of periodic requires two non-parallel directions of translational symmetry. Is it possible to create a tiling that has translational symmetry in only one direction? If so, construct an example of such a tiling.

5. Squares, rectangles, and parallelograms tile the plane. Does every quadrilateral tile the plane? Even non-convex quadrilaterals?



Does this quadrilateral tile?



What about this one?

6. Is there a 5-sided polygon that tiles the plane?

7. Why doesn't a regular pentagon tile the plane?

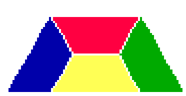
8. Which regular polygons tile the plane?

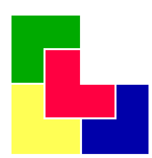
9. Can you find a hexagon that doesn't tile the plane?

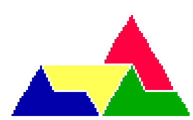
10. Can you find a polygon with more than 6 sides that tiles the plane? Is there a *convex* polygon with more than 6 sides that tiles?

    Definition: A tile **self-replicates** if a finite number of congruent copies of itself fit together to make a larger scaled copy of the tile.

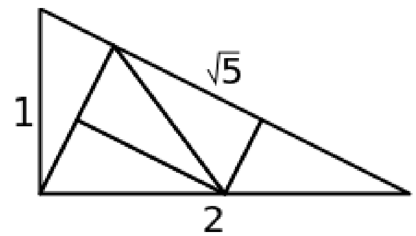
    All triangles self-replicate (why?). Thr L-shape, the isosceles trapezoid, and the "sphinx" also self-replicate.





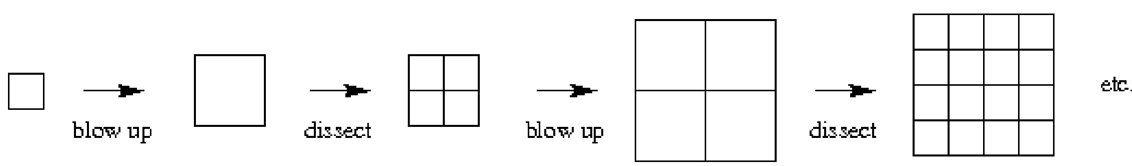


The right triangle with side lengths 1, 2, and $\sqrt{5}$ also self-replicates.

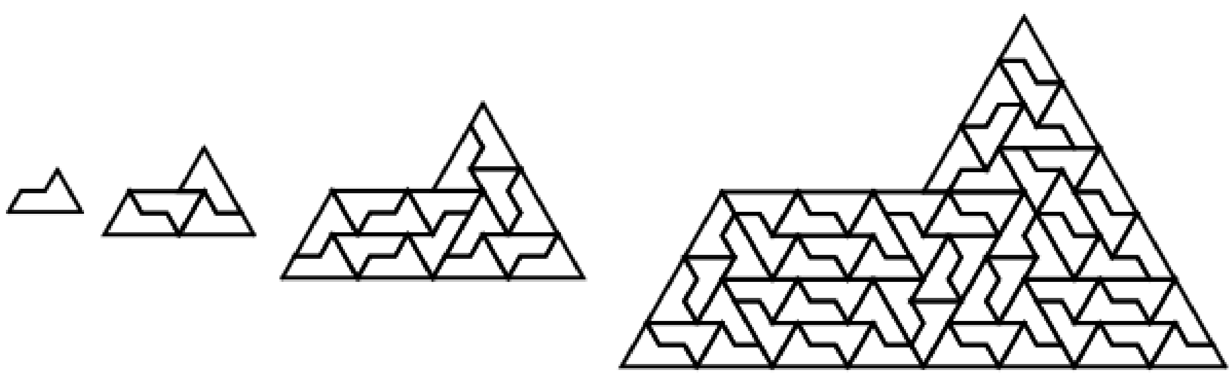


11. Find a figure with the property that just two copies of itself fit together to produce a scaled version of that figure.
12. The L-shape shown above can be divided into 4 copies of itself. Show that it can also be divided into 9, 16, and 25 copies of itself. Can it be divided into a non-square number of copies?

Self-replicating tiles can be used to form tilings of the plane by inflating and subdividing repeatedly. For example, four copies of a square stack together to make a larger square. Repeatedly inflating and subdividing a square (forever) produces the standard square tiling of the plane.



Here is another example using the sphinx.



13. Prove that the self-replicating tiling using the iscosceles trapezoid is non-periodic.

14. Is the self-replicating tessellation using the $1 - 2 - \sqrt{5}$ right triangle non-periodic?

15. What about the tiling using the L-shape?

16. The sphinx?