Sequences and Finite Differences

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For example, given the sequence $a_n = \{1, 4, 7, 10, 13, \ldots\}$, we could make the sequence of differences by subtracting pairs of consecutive terms. [Usually set brackets {} denote a list that's not in order, but they are also used by convention for sequences even though the terms do go in order. Sorry!]

With D representing the "take the difference" operation, because $a_0 = 1$ and $a_1 = 4$, we can write $Da_0 = a_1 - a_0 = 4 - 1 = 3$.
Similarly $Da_1 = a_2 - a_1 = 7 - 4 = 3$. And so on: $Da_n = 3$ for all n. The sequence of first differences is constant.

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1     4     7     10     13     16
   3     3     3      3      3
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Sometimes you have to take the difference of the differences before things get nice. For example, with $a_n = \{0, 1, 4, 9, 16, 25, \ldots\}$, then $Da_n = \{1, 3, 5, 7, 9, \ldots\}$ and then the second differences $D^2a_n = \{2, 2, 2, 2, \ldots\}$. And of course D^3 and any subsequent D's will be all zeros, so we can ignore them.

A nice way to illustrate those differences would be something like

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0     1     4     9     16     25
   1     3     5     7      9
      2     2     2     2
         0     0     0
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perhaps with some … in there to show that it keeps going.

If we know that the second differences are constant, it turns out that the sequence must be a quadratic (parabola). Can we prove that? One way to see that is to start with the formula $a_n = an^2 + bn + c$ and see where the differences lead.
Of course $0^2 = 0$ and so some of the first few terms of the sequence simplify a lot.

$$\begin{array}{ccccccccc} c & & a+b+c & & 4a+2b+c & & 9a+3b+c & & 16a+4b+c \\ & a+b & & 3a+b & & 5a+b & & 7a+b & \\ & & 2a & & 2a & & 2a & & \\ & & & 0 & & 0 & & & \end{array}$$

Now you can take any parabola you might have and determine an equation for it.

1. How can we find a formula for the sum of squares (or the number of cubical building blocks it takes to make a square pyramid)?

2. How do we know that the leading term will be 1/3 n^3? Does that help simplify things at all?

Another approach you can use to understand these sequences, for example n^2 from above, that if you know just a_0 and Da_0 and that D^2 is a constant 2, then you can reconstruct the whole sequence. Here we'll represent that "difference diagonal", otherwise known as the "inverse binomial transform" if you like sounding impressive, as [0, 1, 2], and we'll say n^2 <—> [0, 1, 2], where the arrow there represents the transformation. We'd like to be able to work both ways: that is, given the transformed sequence, find the original sequence. What this "transformation" means is to read off that first diagonal when you write the sequence of differences, or to go the other way, reconstruct the original sequence from that difference diagonal.

Some sequences are really easy. For instance, the sequence {0, 0, 0, 0, …}, which we could just write as 0, has the transformation 0 <—> [0, 0, 0, …]. As with the [0, 1, 2] example, we could leave off the 0's at the end, and just say 0 <—> [].

Not much harder is a constant sequence: 1 <—> [1, 0, 0, …] = [1].

And then there's n: {0, 1, 2, 3, …} <—> [0, 1].

We already got n^2: {0, 1, 4, 9, …} <—> [0, 1, 2].

3. How about n^3? What does it transform to?

Now we'd like to start transforming the other way. The beauty of these transformations (turning a sequence into its difference diagonal, and vice-versa) is that they are *linear*, by which I mean that if we know the transformations of two sequences, the transformation of the sum is the sum of the transformations. Also, if you multiply every number in a sequence by some constant, the transformation is multiplied by the same constant. That is, $T(a_n + b_n) = T(a_n) + T(b_n)$, and $T(k\ a_n) = k\ T(a_n)$, just like with matrices!

4. Show an example of the linearity of this transformation: since n <—> [0, 1] and n^2 <—> [0, 1, 2], verify that $n^2 + n$ <—> [0, 2, 2].
5. Use what you've learned to find the sequence that <—> [0, 0, 1].
6. Find ? <—> [0, 0, 0, 1].

There's another method for finding the sequence that transforms to [0,0,1], too. The sequence, since the second difference is constant, must be a quadratic. But we know the sequence starts {0,0,1}.

7. If you have a quadratic, and $f(0) = 0$, and $f(1) = 0$, what can you conclude about its equation? What form would be easiest to write it in?
8. Now using $f(2) = 1$, finish computing the equation of the quadratic.
9. Similarly find ? <—> [0,0,0,1] by using $f(0) = f(1) = f(2) = 0$ and then $f(3) = 1$, and verify that it matches the answer you got last time you computed this.

Now you're all set to work backwards and forwards too! If you have a sequence whose differences go like [3, 7, 11], you can take 3 [1, 0, 0] + 7 [0, 1, 0] + 11 [0, 0, 1]. Since you know the sequences whose transformations are those three things (namely, the constant 1, the sequence n, and the answer to the problem above which is $n(n-1)/2$, then the formula for this sequence is $3 + 7n + 11\, n(n-1)/2$. To write it out in more detail,

3		$3+7$		$3+7$ $+(7+11)$		$3+7+(7+11)$ $+(7+11+11)$		$3+7+(7+11)$ $+(7+11+11)$ $+(7+11+11+11)$		...
	7		$7+11$		$7+11+11$		$7+11+11+11$		...	
		11		11		11		11		
			0		0		0			

10. Find a formula for the sequence 7, 7, 9, 13, 19, 27, 37, …
11. 0, 1, 5, 14, 30, 55, …?
12. 10, 12, 28, 70, 150, 280, 472, …?
13. 1, 8, 35, 111, 287, 644, 1302, 2430, 4257, 7084, 11297, 17381, …?
14. $mn + b$ <—> ?
15. $an^2 + bn + c$ <—> ?
16. Can you find a general pattern for what <—> [0, 0, …, 0, 1] (in terms of how many 0s are there)?
17. 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, …?
18. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …?

Now, instead of D for going "down a row" into the differences of the sequences, we'll use S to go "up a row", summing the sequence. So for example if you have the sequence of all 1s, we'll write S1 = n to say that the sum of 1, 1, 1, 1, 1 is 0 (written above and just a bit to the left of the first 1), 1, 2, 3, 4, 5. Note that just like the differences will have one fewer term than the sequence, the sums will have one more term. And, somewhat arbitrarily, we'll choose to start with $Sa_0 = 0$ always, $Sa_1 = Sa_0 + a_1$, and in general $Sa_n = Sa_{n-1} + a_n$,

19. Can we find a general formula for the sum of n^{th} powers using this?

20. Now write the first five terms of the sequence whose difference diagonal is [17,8,8,6]. Can you sum this sequence too?

21. What can you say about 2^n? $n2^n$? The sums of these sequences?

22. In what ways are "summing" and "differencing" inverses of each other? In what ways are they not?