VECTORS AND COORDINATES IN 2D, 3D AND BEYOND

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1) A vector is a directed line segment. Notation can be either **a**, $\vec{a}$ or $\vec{AB}$. A vector's length or magnitude can be denoted $|\vec{a}|$, $|\vec{AB}|$, a, AB.

2) Two vectors are *equal* if they have the same length and direction. $\vec{AB} = \vec{CD}$ iff $ABDC$ - parallelogram (note the order of vertices).

3) Zero vector is the one whose tip is same as its tail: $\vec{AA} = \vec{0}$.

4) Vectors are called *collinear* if they lie on same or parallel lines, denoted $\vec{a}||\vec{b}$.

5) Vectors are called *coplanar* if they are located (can be moved) in the same plane.

6) **Vector addition** is done according to *triangle rule*: to the tip of the first vector attach the tail of the second vector. The sum is the vector leading from the tail of the first vector to the tip of the second.

Parallelogram rule: a sum of two vectors originating from the same point is the diagonal of the parallelogram that they span.

7) **Vector subtraction**: $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$. Alternatively, place the two in the same point, and connect their tips, the direction is from the tip of the second vector to the tip of the first.

8) *Triangle inequality*: $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$, with equality taking place iff $\vec{a} \uparrow\uparrow \vec{b}$.

9) **Scalar multiplication**: $\vec{b} = k\vec{a}$ is the same direction as $\vec{a}$: $\vec{b} \uparrow\uparrow \vec{a}$ if $k > 0$, $\vec{b} \uparrow\downarrow \vec{a}$ if $k < 0$, with its magnitude $|\vec{b}| = |k| \cdot |\vec{a}|$. If $k = 0$, then $\vec{b} = \vec{0}$.

Problems.

1. In $\triangle ABC$, E is the midpoint of AB, K is the midpoint of BC, $\vec{m} = \vec{AB}$, and $\vec{n}$ goes from A to C. Express $\vec{BC}$, $\vec{AK}$, $\vec{BE}$, $\vec{EK}$ through $\vec{m}$ and $\vec{n}$.

2. In $\triangle ABC$, K and M are the midpoints of AC and BC, respectively. $\vec{m} = \vec{AB}$, with $\vec{n}$ going from A to K. Express $\vec{BC}$, $\vec{BK}$, $\vec{MK}$, $\vec{AM}$ through $\vec{m}$ and $\vec{n}$.

3. Prove the following: a) $\vec{MA} - \vec{MB} = \vec{BA}$; b) $\vec{XY} + \vec{ZX} + \vec{YZ} = \vec{0}$; c) $(\vec{XY} - \vec{XZ}) + \vec{YZ} = \vec{0}$; d) $(\vec{ZY} - \vec{XY}) - \vec{ZX} = \vec{0}$.

4. Draw pairwise noncollinear $\vec{a}$, $\vec{b}$ and $\vec{c}$. Construct: a) $2\vec{a} + 3\vec{b} - 4\vec{c}$; b) $\frac{1}{2}\vec{a} - \vec{b} + \frac{1}{3}\vec{c}$.

5. Simplify: a) $(\vec{AB} + \vec{BC} - \vec{MC}) + (\vec{MD} - \vec{KD})$; b) $(\vec{CB} + \vec{AC} + \vec{BD}) - (\vec{MK} + \vec{KD})$;

6. Diagonals of parallelogram $ABCD$ intersect in O. Use vectors $\vec{m} = \vec{AB}$ and $\vec{n} = \vec{AD}$ to express the following: $\vec{DC} + \vec{CB}$, $\vec{BO} + \vec{OC}$, $\vec{BO} - \vec{OC}$, $\vec{BA} - \vec{DA}$.

7. A point N on the side BC of $\triangle ABC$ is such that $BN = 2\,NC$. Express vector $\vec{AN}$ through vectors $\vec{a} = \vec{BA}$ and $\vec{b} = \vec{BC}$.

Date: October 20 2014.

A vector formula for the midpoint M of AB: $\vec{OM} = \frac{1}{2}(\vec{OA} + \vec{OB})$, where O is an arbitrary point.

Center mass M of $\triangle ABC$ is: $\vec{OM} = \frac{1}{3}(\vec{OA} + \vec{OB} + \vec{OC})$, where O is arbitrary.

Two nonzero vectors $\vec{a}$ and $\vec{b}$ are collinear iff $\vec{a} = k \cdot \vec{b}$, for some scalar k.

Problems.

8. M and M_1 are respective midpoints of AB and A_1B_1. Prove:

$$\vec{MM_1} = \frac{1}{2}(\vec{AA_1} + \vec{BB_1}).$$

9. Given $\triangle ABC$ and point M, it's known that:

$$\vec{MA} + \vec{MB} + \vec{MC} = 0.$$

Prove that M is the point of intersection of the medians in $\triangle ABC$.

10. Point C divides line segment AB in the ratio $m : n$, counting from A. O is an arbitrary point on the plane. Express $\vec{OC}$ through $\vec{OA}$ and $\vec{OB}$.

11. In a circle centered at O two perpendicular chords intersect in M. Prove:

$$\vec{OM} = \frac{1}{2}(\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD}).$$

Hint. Consider vectors $\vec{OP}$ and $\vec{OQ}$, where P and Q are the chords' midpoints.

12. M, K, N and L are midpoints of AB, BC, CD and DE in a (not necessarily convex) 5-gon $ABCDE$, P and Q are midpoints of MN and KL. Prove using vectors that PQ is one quarter of AE in length and is parallel to it.

13. A and B are on different sides of line l with respective distances of a and b to l. What is the distance from midpoint of AB to l?

14. Diagonals of parallelogram $ABCD$ intersect in M, O is arbitrary. Prove:

$$\vec{OM} = \frac{1}{4}(\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD}).$$

15. Use vectors to prove that the line through midpoints of a trapezoid's bases passes through the point of intersection of its sides.

16. In a quadrilateral, a line connecting two opposite midpoints passes through the point of intersection of its diagonals. Prove that it is a parallelogram or a trapezoid.

17. Let $\triangle ABC$ be inscribed in circle with center O, and H be the point of intersection of the heights of the triangle. Show:

$$\vec{OH} = \vec{OA} + \vec{OB} + \vec{OC}.$$

References

Most of the material here is adopted from A.N.Savin at Samara Medical-Technical Lyceum, `http://mathguru.ru/mtl/geometry9/`.