National Bulgarian Mathematical Olympiad 1997
Regional Round

(1) Find all positive integers a, b, and c such that the roots of the equations

$$x^2 - 2ax + b = 0$$
$$x^2 - 2bx + c = 0$$
$$x^2 - 2cx + a = 0$$

are positive integers.

(2) Let the convex quadrilateral $ABCD$ be inscribed in a circle, $F = AC \cap BD$ and $E = AD \cap BC$. If M and N are the midpoints of AB and CD, prove that

$$\frac{MN}{EF} = \frac{1}{2}\left|\frac{AB}{CD} - \frac{CD}{AB}\right|.$$

(3) Prove that the equation

$$x^2 + y^2 + z^2 + 3(x + y + z) + 5 = 0$$

has no solutions in rational numbers.

(4) Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = f\left(x^2 + \frac{1}{4}\right)$ for all $x \in \mathbb{R}$.

(5) Let K_1 and K_2 be unit squares with centers M and N such that $MN = 4$, two of the sides of K_1 are parallel to MN and one of the diagonals of K_2 lies on the line MN. Find the locus of midpoints of the segments XY where X and Y are interior points of k_1 and k_2, respectively.

(6) Find the number of all non-empty subsets of the set $S_n = \{1, 2, \ldots, n\}$ which do not contain two consecutive integers.

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Final Round

(1) Consider the polynomials $P_n(x) = \sum_{i=0}^{k} \binom{n}{3i+2} x^i$ where $n \geq 2$ and $k = \lfloor \frac{n-2}{3} \rfloor$.

(a) Prove that $P_{n+3}(x) = 3P_{n+2}(x) - 3P_{n+1}(x) + (x+1)P_n(x)$.

(b) Find all integers a such that $P_n(a^3)$ is divisible by $3^{\lfloor \frac{n-2}{3} \rfloor}$ for all $n \geq 2$.

(2) Let M be the centroid of $\triangle ABC$. Prove the inequality

$$\sin \angle CAM + \sin \angle CBM \leq \frac{2}{\sqrt{3}}.$$

(a) if the circumcircle of $\triangle AMC$ is tangent to the line AB;

(b) for any $\triangle ABC$.

(3) Let n and m be positive integers and $m + i = a_i b_i^2$ for $i = 1, 2, ..., n$, where a_i and b_i are positive integers and a_i is square-free. Find all n for which there exists m such that $a_1 + a_2 + \cdots + a_n = 12$.

(4) Let a, b, and c be positive numbers such that $abc = 1$. Prove the inequality

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}.$$

(5) Let BM and CN be the angle bisectors in $\triangle ABC$ and let ray $MN^{\rightarrow}$ intersect the circumcircle of $\triangle ABC$ at point D. Prove that

$$\frac{1}{BD} = \frac{1}{AD} + \frac{1}{CD}.$$

(6) Let X be a set of $n+1$ elements, $n \geq 2$. The ordered n-tuples $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ consisting of distinct elements of X are called "disjoint" if there exist distinct indices i and j such that $a_i = b_j$. Find the maximal number of n-tuples an two of which are "disjoint".