# Mathematical Games

**Tatiana Shubin**

shubin@math.sjsu.edu

In all the games described below there are two players, Alice and Bob, and Alice always plays first. The problem is to decide which one of the two players has a winning strategy (and, of course, to describe this strategy).

An answer to the question "Which player has a winning strategy?" must include a detailed description of such strategy, i.e., you have to explain what the winning player should do so that this player wins REGARDLESS of his opponent's moves.

To *solve* a game means to find a winning or a non-losing strategy for one of the players.

1. (a) There are 25 matches on a table. On each turn, a player can take any number of matches between 1 and 4. The player that takes the last match wins.
   (b) Same game as above but it starts with 24 matches.
   (c) Same game again, only the initial number of matches is N.

2. (a) Now there are two piles of matches, one pile with 10 matches and another one with 7. On each turn, a player can take any number of matches from either one of the two piles. The player who takes the last match wins.
   (b) What will happen if the numbers of matches in the piles are $m$ and $n$?

3. (a) Alice and Bob want to produce a 20-digit number, writing one digit at a time from left to right. Alice wins if the number they get is not divisible by 7; Bob wins if the number is divisible by 7.
   (b) What will happen if 7 is replaced by 13 in the previous game?

4. Given a convex $n$-gon, the players take turns drawing diagonals that do not intersect those diagonals that have already been drawn. The player unable to draw a diagonal loses.

5. There are 25 matches in a pile. A player can take 1, 2, or 4 matches on each turn. The player who cannot continue (no more matches left) loses.

6. On one square of an 8 by 8 chessboard there is a "lame tower" that can move either to the left or down through any number of squares. Alice and Bob take turns moving the tower. The player unable to move the tower loses. (Consider various initial positions of the tower.)

7. There are two piles of matches; one pile contains 10 matches while the other contains 7. A player can take one match from the first pile, or one match from the second pile, or one match from each of the two piles. The player unable to move loses.

8. At the start of the game, there is a number 60 written on the board. On each turn, a player can reduce the number that is currently on the board by any of its positive divisors. If the resulting number is a 0, the player loses.

9. Alice calls out any integer between 2 and 9, Bob multiplies it by any integer between 2 and 9, then Alice multiplies the new number by any integer between 2 and 9, and so on. The player who first gets a number bigger than 1000 wins.

10. Players take turns putting pennies on a round table. The pennies cannot overlap; they can overhang the table but should not fall off. The player unable to place a penny loses.

11. (a) There are several minuses written along a line. A player replaces either one minus by a plus or two adjacent minuses by two pluses. The player who replaces the last minus wins.
    (b) Same game as above, only the minuses are written around a circle.

12. There are nine cards on a table labeled by numbers 1 trough 9. Alice and Bob take turns choosing one card. The player that has collected a set of cards with the property that the sum of numbers on three cards out of total set is 15 wins. There's a tie if none of the players has a set of cards with this property at the end of the game. Does any of the players have a winning strategy? A non-losing strategy?

13. Players start with one pile of pebbles. On each move, a player must split one pile into two nonempty piles in such a way that all resulting piles have different number of pebbles. The player unable to make a move loses.
    (i) After the first move, the piles contain 5 and 11 pebbles. Find a winning strategy for Bob.
    (ii) After the first move, the piles contain 5 and 11 pebbles. Give an example of a bad move after which Bob will necessarily lose.
    (iii) Which player has a winning strategy if they start with 11 pebbles?
    (iv) Which player has a winning strategy if they start with 22 pebbles?
    (v) Can you solve the game in general?

14. In a box, there are
    (a) 57 candies;
    (b) 50 candies;
    (c) 1000 candies.
    (d) $N > 1$ candies.

On each turn, a player can take any amount of candy subject to the following two conditions.

1. The first player cannot take all the candy.
2. A player cannot take more candy than his opponent has just taken.

The player who takes the last candy wins. Which of the players has a winning strategy?

15. Suppose we start with N integers: 1, 2, 3, … , N. On each turn, a player circles one of the numbers in such a way that all circled numbers are pairwise relatively prime. No number can be circled twice. The player unable to complete a turn loses.
Which player has a winning strategy if:
    (a) N = 10;
    (b) N = 12;
    (c) N = 15;
    (d) N = 30;
    (e) N is any positive integer.