Marin Math Circle   Jan. 27, 2010

Survival Combinatorics   *Instructor: Paul Zeitz, zeitz@usfca.edu*

Not So Hard

1 A taqueria sells burritos with the following fillings: pork, grilled chicken, chicken mole, and beef. Burritos come either small, medium, or large, with or without cheese, and with or without guacamole. How many different burritos can be ordered?

2 How many subsets does a set with 8 elements have, including the empty set and the set itself?

3 Given a pool of 30 students, how many ways can we choose a 3-person government consisting of a president, vice-president, and treasurer?

4 In Exercise 3, we tacitly assumed that no one could hold more than one office. Verify that if we allowed this, then the answer would be 30^3.

5 10 boys and 9 girls sit in a row of 19 seats. How many ways can this be done if

   (a) All the boys sit next to each other and all the girls sit next to each other?

   (b) The children sit so that each person has only neighbors of the opposite sex.

6 How many ways can you choose a team from 11 people, where the team must have at least one person and the team must have a designated captain?

7 (a) In a traditional village, there are 10 boys and 10 girls. The village matchmaker arranges all the marriages. In how many ways can she pair off the 20 children? Assume (the village is traditional) that all marriages are heterosexual (i.e., a marriage is a union of a male and a female; male-male and female-female unions are not allowed).

   (b) In a not-so-traditional village, there are 10 boys and 10 girls. The village matchmaker arranges all the marriages. In how many ways can she pair off the 20 children, if homosexual marriages (male-male or female-female) as well as heterosexual marriages are allowed?

8 How many even 3-digit numbers have no repeating digits?

9 An *n*-bit string is an *n*-digit binary number, i.e., a string of just zeros and ones. How many 10-bit strings contain *exactly* 5 consecutive zeros (no more, no less)? For example, we would not count 0000000111 (too many consecutive zeros), but we would count 1110000011 and 0011000001.

10 Three different flavors of pie are available, and 7 children are each given a slice of pie in such a way that at least two children get different flavors. How many ways can this be done?

11 How many subsets of the set $\{1,2,3,4,\ldots,30\}$ have the property that the sum of the elements of the subset is greater than 232?

12 How many 10-bit strings have *exactly* 4 zeros?

13 Suppose that the n vertices of a polygon are arranged on the circumference of a circle so that no three diagonals intersect in the same point.

   (a) How many diagonals that intersect in the interior of the circle are there?

   (b) How many intersection points do these diagonals have?

14 How many *strictly increasing* sequences of positive integers begin with 1 and end with 1000?

15 How many ways can the positive integer n can be written as an ordered sum of at least one positive integer? For example,

$$4 = 1+3 = 3+1 = 2+2 = 1+1+2 = 1+2+1 = 2+1+1 = 1+1+1+1,$$

so when $n = 4$, there are 8 such *ordered partitions*.

Not So Easy

1 How many ways can 10 people form two teams of 5?

2 How many ways can 7 dogs consume 10 dog biscuits? The biscuits are indistinguishable, the dogs are distinguishable. Dogs do not share.

3 Decide whether there exist 10,000 10-digit numbers divisible by 7, all of which can be obtained from one another by a reordering of their digits.

4 Imagine a piece of graph paper. Starting at the origin draw a path to the point $(10,10)$, that stays on the grid lines (which are one unit apart) and has a total length of 20. For example, one path is to go from $(0,0)$ to $(0,7)$ to $(4,7)$ to $(4,10)$ to $(10,10)$. Another path goes from $(0,0)$ to $(10,0)$ to $(10,10)$. How many possible different paths are there?

5   (a) In how many ways can two squares be selected from an 8-by-8 chessboard so that they are not in the same row or the same column?

   (b) In how many ways can four squares, not all in the same row or column, be selected from an 8-by-8 chessboard to form a rectangle?

6 Prove the following facts about binomial coefficients.

   (a) For any positive integer n, with $r \leq n$,

$$\binom{n}{r} = \binom{n}{n-r}.$$

   (b) Because

$$\binom{n}{n} = \binom{n}{0} = 1,$$

   the only logical value for $0!$ is 1.

   (c) For any positive integer n, with $r \leq n$,

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}.$$

   (d) For any positive integer n,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

7 (a) Suppose we have three different toys and we want to give them away to two girls and one boy (one toy per child). The children will be selected from 4 boys and 6 girls. In how many ways can this be done?

   (b) Suppose again that we have three different toys and we want to give them away (one toy per kid) to three children selected from a pool of 4 boys and 6 girls, but now we require that at least two boys get a toy. In how many ways can this be done?

8 How many different ordered triples (a,b,c) of non-negative integers are there such that $a+b+c = 50$? What if the three integers had to be positive?

9 Ten children order ice cream cones at a store featuring 31 flavors. How many orders are possible in which at least two children get the same flavor?

10 There are 10 adjacent parking spaces in the parking lot. When you arrive in your new Rolls Royce, there are already seven cars in the lot. What is the probability that you can find two adjacent unoccupied spaces for your Rolls? Generalize.

11 Let S be a set with n elements. In how many different ways can one select two not necessarily distinct subsets of S so that the union of the two subsets is S? The order of selection does not matter; for example, the pair of subsets $\{a,c\}, \{b,c,d,e,f\}$ represents the same selection as the pair $\{b,c,d,e,f\}, \{a,c\}$.

12 Ten dogs encounter 8 biscuits. Dogs do not share biscuits! How many different ways can the biscuits can be consumed

    (a) if we assume that the dogs are distinguishable, but the biscuits are not;

    (b) if we assume that the dogs and the biscuits are distinguishable (for example, each biscuit is a different flavor).

    (c) if we assume that neither the dogs nor the biscuits are distinguishable? (We are able to distinguish dogs from biscuits, however. The answer is *not* 1!)

13 A random number generator randomly generates the integers $1, 2, \ldots, 9$ with equal probability. Find the probability that after n numbers are generated, the product is a multiple of 10.

14 Decide whether there exist 10,000 ten-digit numbers divisible by seven, all of which can be obtained from one another by a reordering of their digits.

15 Let $t(n)$ be the maximum number of different areas that you can divide a circle into when you place n points on the circumference and draw all the possible line segments connecting the points. It is easy to check (verify!) that

$$t(1) = 1, \quad t(2) = 2, \quad t(3) = 4, \quad t(4) = 8, \quad t(5) = 16.$$

The conjecture that $t(n) = 2^{n-1}$ is practically inescapable. Yet $t(6)$ is equal to 31, not 32 (again, verify!), so something else is going on. Anyway, can you deduce the correct formula for $t(n)$?