MEDIANS SURRENDER AT THE OLYMPICS

Geometry at the Bay Area Mathematical Olympiad[1]

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Abstract: Have you heard of the expression "*center of mass of* $\triangle ABC$"? Very likely you have! If M is the midpoint of side BC, then segment AM is called a *median* of the triangle. If you draw the three medians very carefully, you will discover that they meet at some point G. If you hang your triangle (conveniently made of cardboard!) on a string from this point G, you will discover that the triangle stays horizontal to the floor! This is why point G is called the *centroid* (or *center of mass*) of $\triangle ABC$. Try it! In this session we will see how the three medians and the centroid challenge students in puzzling geometry problems from the *Bay Area Mathematical Olympiad (BAMO)*, only to surrender to students' creative solutions via geometric transformation, extra constructions, and dust-covered century-old theorems!

Medians and Centroids

(1) **(BAMO '00)** Let ABC be a triangle with D the midpoint of side AB, E the midpoint of side BC, and F the midpoint of side AC. Let k_1 be the circle passing through points A, D, and F; let k_2 be the circle passing through points B, E, and D; and let k_3 be the circle passing through points C, F, and E. Prove that circles k_1, k_2, and k_3 intersect in a point.

(2) **(BAMO '05)** If two medians in a triangle are equal in length, prove that the triangle is isosceles.

(3) **(BAMO '06)** In $\triangle ABC$, choose point A_1 on side BC, point B_1 on side CA, and point C_1 on side AB in such a way that the three segments AA_1, BB_1, and CC_1 intersect in one point P. Prove that P is the centroid of $\triangle ABC$ if and only if P is the centroid of $\triangle A_1B_1C_1$.

Geometry on the Circle

(4) **(BAMO '99)** Let C be a circle in the xy-plane with center on the y-axis and passing through $A = (0, a)$ and $B = (0, b)$ with $0 < a < b$. Let P be any other point on the circle, let Q be the intersection on the line through P and A with the x-axis, and let $O = (0, 0)$. Prove that $\angle BQP = \angle BOP$.

(5) **(BAMO '99, shortlisted IMO '98)** Let $ABCD$ be a cyclic quadrilateral (i.e., it can be inscribed in a circle). Let E and F be variable points o the sides AB and CD, respectively, such that $AE/EB = CF/FD$. Let P be the point on segment EF such that $PE/PF = AB/CD$. Prove that the ratio between the areas of $\triangle APD$ and $\triangle BPC$ does not depend on the choice of E and F.

(6) **(BAMO '02)** Let ABC be a right triangle with right angle at B. Let $ACDE$ be a square drawn exterior to $\triangle ABC$. If M is the center of the square, find the measure of $\angle MBC$.

Projective Geometry?

(7) **(BAMO '01)** Let $JHIZ$ be a rectangle, and let A and C be points on sides ZI and ZJ, respectively. The perpendicular from A to CH intersects line HI in X, and the perpendicular from C to AH intersects line HJ in Y. Prove that X, Y and Z are collinear (i.e., lie on the same line).

(8) **(BAMO '06)** In $\triangle ABC$, choose point A_1 on side BC, point B_1 on side CA, and point C_1 on side AB in such a way that the three segments AA_1, BB_1, and CC_1 intersect in one point P. Prove that P is the centroid of $\triangle ABC$ if and only if P is the centroid of $\triangle A_1B_1C_1$.

[1]At the Marin Math Circle at Dominican University, October 13 2010, Wednesday, 6:30-8pm.

ATTACKING PLANE GEOMETRY:
WITH BARE HANDS OR WITH MATHEMATICAL ARMOR?

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Abstract: In the classic book of "Alice in Wonderland" many strange things happen that are left unexplained by the mathematician author Louis Carol. Similarly, in this math circle session at UT Dallas, reflections will "mystically" become rotations and rotations will turn into translations! Is this possible and mathematically sound? Come to this talk to find out what happened just a month ago at the Bay Area Math Olympiad and how three different brilliant solutions to the same geometry problem were created by student participants.

Transformations in the Plane

(9) **(BAMO '07)** In $\triangle ABC$, D and E are two points inside side BC such that $BD = CE$ and $\angle BAD = \angle CAE$. Prove that $\triangle ABC$ is isosceles.

(10) **(BAMO '10)** Acute $\triangle ABC$ has $\angle BAC < 45°$. Point D lies in the interior of $\triangle ABC$ so that $BD = CD$ and $\angle BDC = 4\angle BAC$. Point E is the reflection of C across line AB, and point F is the reflection of B across line AC. Prove that lines AD and EF are perpendicular.

Geometric Constructions

(11) **(BAMO '04)** A given line passes through the center O of a circle. The line intersects the circle at points A and B. Point P lies in the exterior of the circle and does not lie on the line AB. Using only an unmarked straightedge, construct a line through P, perpendicular to the line AB. Give complete instructions for the construction and prove that it works.

(12) **(BAMO '09)** Seven congruent line segments are connected together at their endpoints as shown in the figure below at the left. By Raising point E, the linkage can be made taller, as shown in the figure below and to the right. Continuing to raise E in this manner, it is possible to use the linkage to make A, C, F and E collinear, while simultaneously making B, G, D, and E collinear, thereby constructing a new triangle ABE.

Prove that a regular polygon with center E can be formed from a number of copies of this new triangle ABE, joined together at point E and without overlapping interiors. Also find the number of sides of this polygon and justify your answer.

Geometric Equalities and Inequalities

(13) **(BAMO '03)** Let $ABCD$ be a square, and let E be an internal point on side AD. Let F be the foot of the perpendicular from B to CE. Suppose G is a point such that $BG = FG$, and the line through G parallel to BC passes through the midpoint of EF. Prove that $AC < 2 \cdot FG$.

(14) **(BAMO '08)** Point D lies inside $\triangle ABC$. If A_1, B_1, and C_1 are the second intersection points of the lines AD, BD, and CD with the circles circumscribed about $\triangle BDC$, $\triangle CDA$, and $\triangle ADB$, prove that

$$\frac{AD}{AA_1} + \frac{BD}{BB_1} + \frac{CD}{CC_1} = 1\cdot$$