Digit Sums and Beyond

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Joshua Zucker, joshua.zucker@stanfordalumni.org

1. Let $A = 4444^{4444}$. Let B be the sum of the digits of A. Let C be the sum of the digits of B. Let D be the sum of the digits of C. What is the value of D? Can you figure out values for any of the other variables (without a computer)?

2. Is there a nine-digit number that uses each digit 1 through 9 exactly once, such that the first digit makes a one-digit number divisible by 1, the first two digits make a two-digit number divisible by 2, the first three digits make a three-digit number divisible by 3, and so on?

3. I want to make a total of 100 by summing numbers whose only digits are 0 and 8. Can it be done? What is the fewest numbers that can be used?

4. Does every number have a multiple that consists only of the digits 0 and 8? If you can prove that one must exist, then can you find such a multiple for an arbitrary number, like say 12 or 37 or 72?

5. What positive integer is doubled when you take its last digit and move it to the front? For example, 526 is not quite exactly double 265.

6. What variations of the previous question can you imagine? Do they all have solutions? Unique solutions?

7. The number 2^{29} is a nine-digit number that, interestingly enough, turns out to have nine different digits. Which digit is missing?

8. Is there a number whose ten digits are ABCDEFGHIJ so that A tells you how many 0s are in the number, B tells you how many 1s, C tells you how many 2s, and so on?

9. What are the last two digits of 9^8^7^6^5^4^3^2^1? (Expressions like these are called "towers of powers" and you need to evaluate them from right to left, so here the 2^1 should be computed first, then 3 to that power, then 4 to that power, and so on.)

10. How many ways can you use the digits 1 through 9 to make two numbers, one of which is the square of the other? For instance the square of 569 is 323761 but that has two 3s and two 6s and no 4 or 8.

11. (Thanks to Stan Wagon) A positive integer is called *balanced* if the number of its digits is equal to the number of its distinct prime factors. For instance, 15 is balanced because it has two digits and two prime factors, but 49 is not. What is the largest balanced number?

12. (Another one from Stan Wagon) For what numbers n is the number of 1s among the digits of the numbers 1 through n equal to n? For instance, this works for 1 because there's only 1 number and it has one 1. It doesn't work for 11 because up to 11 we find four 1s and need eleven of them. (Are there any solutions besides $n = 1$? Is there a largest solution?)

13. (Thanks to Stan Wagon and *Puzzles 101*, by Nobuyuki Yoshigahara (AK Peters, Inc.)) Given any two digits, look at all the positive integers made from those two digits. Find the smallest possible number made that way which is divisible by both digits. For example, for 2 and 4 the answer is 24. For 2 and 5 there is no answer (why?). For 3 and 4 the answer is 3444. For which two digits is the answer largest?

14. Take any odd power of 5, say 125 for example. Add 1. Divide by 6. What are the last two digits? Is this a coincidence? Does the same kind of thing work for any numbers other than the ones given here? How about in other bases?

15. (Another from Stan Wagon, thanks!) Find all numbers n such that the sum of the digits of n, $2n$, $3n$, ..., nn are all equal. (Are there any such numbers greater than 1?)

16. (And another!) Is it true that for any sequence of at most 10 different letters, like MATH or MATHEMATICS or INTERESTING, there is a way to assign different digits to the letters such that the resulting number is divisible by 7?

17. (1991 Leningrad Math Olympiad) Is it possible to reduce any integer to a one-digit number by using a combination of rule (A) which lets you multiply your integer by any positive integer, and rule (B) which deletes all the zero digits from the number? (Hint: Perhaps one of the earlier problems in this set will be helpful.)