Marin Math Circle **March 30, 2011**
PROBABILITY **Instructor: Paul Zeitz**
(zeitz@usfca.edu)

Fun with Dice

1 *Thirty-six scenarios*. When two dice are rolled, there are 36 different outcomes, because $36 = 6 \times 6$. Each outcome is equally likely. This allows us to compute probabilities easily. Use this table to compute the various sums that can occur. I filled in the first few cells.

	1	2	3	4	5	6
1	2	3	4			
2		4				
3						
4						
5						
6						12

(a) Verify that the probability of rolling two dice and getting a sum of 2 is $1/36$.

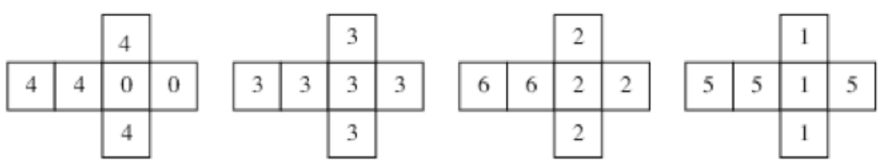
(b) What is the probability of rolling two dice and getting a sum of 5?

(c) What is the most likely sum, and what is its probability?

(d) Test these probabilities by experiment. For example, suppose you want to test for a sum of 12, which should also have a probability of $1/36$. You'd expect to see this sum happen only once every 36 or so rolls. Experiment!

2 *Non-transitive dice*. Construct the following set of four dice.

A B C D



(a) Suppose one person tosses the A die, and the other tosses the B die. The winner is the one whose die has the bigger number. What is the probability that A wins? You will need to make a 6×6 table like this. I filled in a few cells, saying who the winner is.

A vs B	0	0	4	4	4	4
3	B	B	A			
3						
3						
3						
3						
3						

(b) Work out the probabilities for the other possibilities, such as B vs. C, C vs. D, etc. Here are some tables to help you organize your work.

B vs. C	2	2	2	2	6	6
3	B	B			C	
3						
3						
3						
3						
3						

C vs. D	2	2	2	2	6	6
1	C	C				
1						
1						
5						
5	D					
5						

A vs. C	2	2	2	2	6	6
0						
0						
4						
4						
4						
4						

A vs. D	1	1	1	5	5	5
0						
0						
4						
4						
4						
4						

(c) Now that you have collected data, do you notice something strange about these four dice? Can you construct a "sucker bet" with them?

3 What's the probability of rolling six dice and

(a) getting a sum of 6?
(b) getting a sum of 7?
(c) getting a sum of 10?
(d) having all six numbers be equal?
(e) having all six numbers be different?

4 *How to resolve a Love Triangle*. Akira, Betsy and Cleopatra fight a 3-cornered pistol duel. All three know that Akira's chance of hitting any target is $1/3$, while Betsy *never* misses, and Cleopatra has a 0.5 chance of hitting any target. The way the duel works is that each person is to fire at their choice of target, starting with Akira, and proceeding in alphabetical order (unless someone is hit, in which case they don't shoot), continuing until one person is left unhit. What is Akira's strategy?

Experiment with different strategies, and simulate the shooting using your dice. You can simulate an event with probability $1/3$ by tossing one die and seeing if the the number shown is 1 or 2, say. Likewise, you can come up with ways to simulate a $1/2$ probability event (you don't need a coin, you can still use a die).

Here are some possible strategies for Akira: shoot at Cleopatra first; shoot at Betsy first; try something else. Choose a strategy, and try simulating the duel. See if you can experimentally estimate the probability that Akira survives, using various strategies.

5 On average, how many times must a die be thrown until one gets a 6?

How many times, on average, should one toss a fair die in order to see all 6 possible outcomes?

Problems Involving Expectation, Random Walks, and Others

1 A deck of cards is randomly cut into three piles. I bet that at least one of the cards on the top of a pile is a "face card;" i.e., a Jack, Queen, or King? Do you want to take this bet?

2 *The Classic Birthday Problem.* In a room of 30 people, how likely is it that at least two people have the same birthday?

3 *Another Birthday Problem.* In Klopstockia, all factory workers must work every day of the year, getting no days off, with only one exception: whenever it is a worker's birthday, the factory closes down and everyone gets the day off! Factories are obligated to hire workers randomly, without discriminating on the basis of birthday. In order to maximize output, how many workers should a factory employ?

4 It costs a consumer \$1 to buy a Klopstockia lottery ticket. The buyer then scratches the ticket to see the prize. Compute, to the nearest penny, the expected profit that the state of Klopstockia makes per ticket sold, given the following scenarios for prizes awarded. (The state will make a profit if the expected value of the lottery ticket is *less* than \$1.)

(a)

Prize	\$1	\$10	\$1000
Probability	$\frac{1}{10}$	$\frac{1}{1{,}000}$	$\frac{1}{1{,}000{,}000}$

(b)

Prize	\$1	\$10	\$1000	a free lottery ticket
Probability	$\frac{1}{10}$	$\frac{1}{1{,}000}$	$\frac{1}{1{,}000{,}000}$	$\frac{1}{5}$

5 Two players alternately toss a penny, and the one that first tosses heads wins. What is the probability that

(a) the game never ends?

(b) the first player wins?

(c) the second player wins?

6 *Hint: Expectation is Additive*.

(a) Place n letters at random into n envelopes. What is the average number of letters which get into the correct envelopes?

(b) Shuffle an ordinary deck of 52 playing cards. On the average, how far from the top will the first ace be?

(c) We randomly place c cats and d dogs in a row. On the average, how many pairs of adjacent positions are "heterospecious," i.e. dog-cat or cat-dog?

(d) An urn contains 1000 balls, labeled $1, 2, 3, \ldots, 1000$. You perform the following procedure: mix the balls well, pick a ball, record its number, and then put it back in the urn. If you do this procedure 1000 times, what is the average number of *distinct* integers that you will record?

7 *Finite Random Walks*.

(a) Suppose you are sitting on the number line, at position k, and you flip a fair coin. If it is heads, you move one space to the right. If it is tails, you move one space to the left. You lose if you hit 0 and win if you hit 10. What is the probability that you will win?

(b) Generalize to unfair coins, and of course let 10 be any positive integer.

8 *The Classic Gambler's Ruin Problem*. Two players take turns tossing a fair coin. If the coin is heads, player A gives player B a dollar. If the coin turns up tails, B gives A a dollar. Player A starts with a dollars, and player B starts with b dollars (a and b are non-negative integers). Once a player goes bankrupt (i.e., has zero dollars) the game is over. What is the probability that A goes bankrupt?

What happens if the probabilities are not equal; i.e., what if the probability that the coin is heads is p, for some fixed $0 \leq p \leq 1$.

9 *A Bug on a Cube*. Imagine a bug that crawls along the edges of a cube. The bug does not change directions while traveling on an edge. Two adjacent vertices, F and P, have food and poison, respectively. If the bug reaches either of these vertices, it stops traveling. Whenever the bug reaches one of the other six vertices, it has a choice of three edges on which to travel and it chooses randomly (i.e., with probability of $1/3$ for each choice). For each of these six starting vertices, compute the probability that the bug lives (i.e., reaches F before reaching P).

10 *What a Loser!* You arrive in Las Vegas with \$100 and decide to play roulette, making the same bet each time, until you are either bankrupt or have doubled your money. Which of the following strategies is best?

(a) Making bets of \$1 each time.

(b) Making bets of \$10 each time.

(c) Making a single bet of \$100.

11 *A Gambling "System."* Suppose you are playing a game with a 50% chance of winning each time. You can bet any amount, and if you win, you win twice your bet. If you lose, you lose your bet. In other words, if you bet B dollars, your *profit* is $\pm B$ depending on whether you win or lose. You decide that you will play, stopping as soon as you win, doubling the size of your bet each time. You are guaranteed to make a profit? Right? Use expectation to show that this won't work. What if you triple instead of double?

12 *The St. Petersburg Paradox*. Consider the following game. I will flip a fair coin until it shows up heads. We keep track of the number of flips until this happens. If it happens on the first flip, I'll pay you \$2. If it takes two flips, then I'll pay you \$4. Three flips, \$8, etc. In other words, if it takes n flips until the first head, I will pay you 2^n dollars. Pretty sweet game!

How much is this game worth *to you*? In other words, if a there was a ticket that allowed you to play the game once with me (I flip the coin until it is heads, and pay you the appropriate amount), how much would you pay for the ticket? Clearly, you'd pay at least 1 dollar. In fact, you'd almost certainly pay 2 dollars. How about 3? 4? 5? More?

13 *Infinite Random Walks*.

(a) In a one-dimensional random walk with equal probabilities of moving one unit left or right, what is the probability that you return to your starting point after 10 moves?

(b) Answer the same question, but with a two-dimensional random walk (so now there are four directions you can move, each with equal probability).

(c) After n steps, how far, on average, will a 1-dimensional random walk be from the starting point? How about 2-dimensional? r-dimensional?

(d) What is the probability that a random walk will return to its starting point? Does this depend on the dimension?