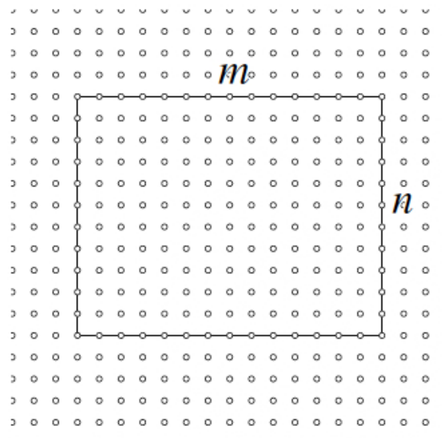
Marin Math Circle: May 11, 2011
Two Proofs of Pick's Theorem

1 A Proof of Pick's Theorem using Induction

(Reference: `http://www.geometer.org/mathcircles/pick.pdf`).

In this series of exercises, you will prove Pick's Theorem using induction.

1. Consider an $m \times n$ lattice-aligned rectangle:



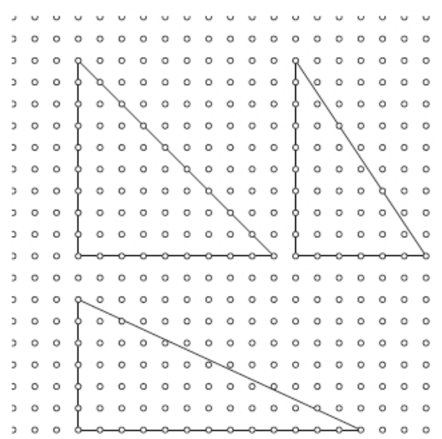
Show that for such a rectangle,

$$I = (m-1)(n-1) \text{ and } B = 2m + 2n.$$

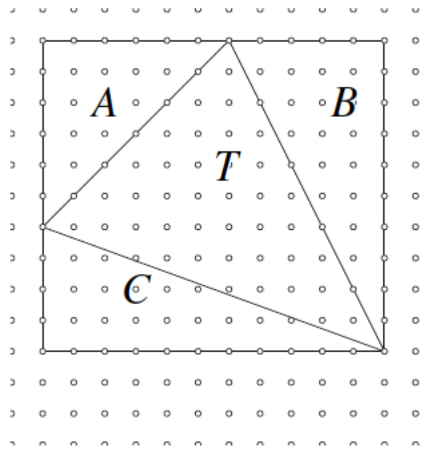
Conclude that

$$A = I + \frac{B}{2} - 1.$$

2. Next, find I and B for a lattice-aligned right triangle with legs m and n. Prove that Pick's Theorem holds for such a triangle.



3. The next step is to show that Pick's Theorem holds for arbitrary triangles. If T is an arbitrary triangle, draw right triangles A, B, C to form a rectangle R, as shown below.



Suppose that triangle A has I_a interior points and B_a boundary points. Use similar notation for triangles B and C. Let I_r and B_r denote the number of interior and boundary points of the rectangle, respectively. We already know that Pick's Theorem holds for A, B, C, R, so we know that

$$\begin{aligned} A(A) &= I_a + \frac{B_a}{2} - 1 \\ A(B) &= I_b + \frac{B_b}{2} - 1 \\ A(C) &= I_c + \frac{B_c}{2} - 1 \\ A(R) &= I_r + \frac{B_r}{2} - 1 \end{aligned}$$

We want to show that

$$A(T) = I_t + \frac{B_t}{2} - 1.$$

We know that

$$A(T) = A(R) - A(A) - A(B) - A(C).$$

Explain why

$$B_r = B_a + B_b + B_c - B_t$$

and

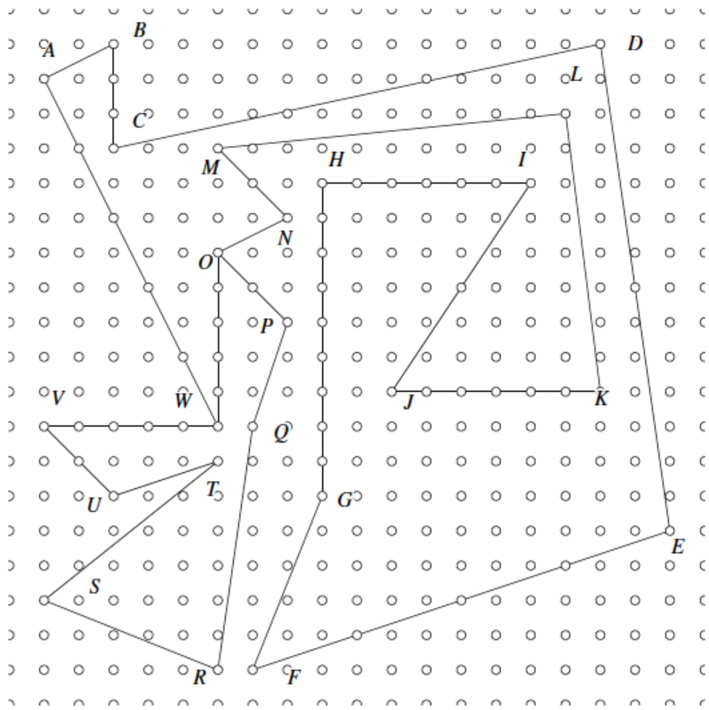
$$I_r = I_a + I_b + I_c + I_t + (B_a + B_b + B_c - B_r) - 3.$$

Use these equations to show that

$$A(T) = I_t + \frac{B_t}{2} - 1.$$

4. So far, we've shown that Pick's Theorem is true for every polygon with 3 sides. To complete the proof that Pick's Theorem is true for any polygon, we'll use induction on the number of sides of the polygon. The base case is $n = 3$ sides, and we've already shown that Pick's Theorem holds for $n = 3$. For the inductive step, assume that Pick's Theorem holds for $n = 3, 4, \ldots, k - 1$ sides. We must now prove that Pick's Theorem holds for $n = k$ sides to complete the induction.

   Suppose that P is a polygon with k sides ($k > 3$). Show that P must have an *interior diagonal* that will split P into 2 smaller polygons. Here's an example. OW is the interior diagonal for this example.



Once we have shown that we can always split a polygon P with k sides into 2 smaller polygons P_1 and P_2 (each with fewer than k sides), the final step is to show that if 2 polygons satisfy Pick's Theorem, then the polygon formed by *attaching* the 2 will also satisfy Pick's Theorem. Since the smaller polygons satisfy Pick's Theorem by the inductive hypothesis, we have

$$A(P) = A(P_1) + A(P_2) = I_1 + \frac{B_1}{2} - 1 + I_2 + \frac{B_2}{2} - 1.$$

Finally, find a relationship between I and I_1, I_2 and between B and B_1, B_2 to conclude that

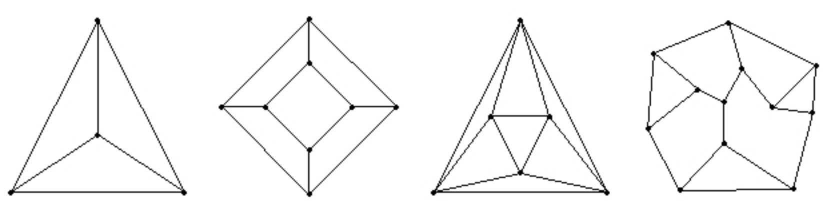
$$A(P) = I + \frac{B}{2} - 1.$$

2 A Proof of Pick's Theorem using Euler's Formula

A **graph** $G = (V, E)$ consists of a V, a nonempty, finite set of *vertices*, and E, a finite set of unordered pairs of distinct elements of V called *edges*. If $\{u, v\}$ is an edge e of G, the edge e is said to connect u and v, and the vertices u and v are the *endpoints of the edge e*. Some examples of graphs are shown below. These graphs are both *planar* (can be drawn in such a way that no 2 edges cross) and connected (there is a path between every pair of distinct vertices of G). Euler's Formula states that if G is a connected, planar graph with n vertices, e edges, and f regions, then

$$v - e + f = 2.$$

(A planar graph splits the plane into *regions*, including one unbounded region "outside" the graph. For example, the graph on the far left below contains 4 regions.)



Can you prove Euler's formula?

1. To prove Pick's Theorem from Euler's Formula, start with a simple lattice polygon P, and show that you can always dissect the polygon into *primitive lattice triangles*. A *primitive lattice triangle* is a triangle that has no lattice points in its interior, and no lattice points other than vertices on its sides.

2. Next, show that the area of a primitive lattice triangle is equal to $\frac{1}{2}$. (Of course, you can't use Pick's Theorem since that's what we're trying to prove! You must use geometry to prove this result.)

3. Once you've proved these results, it's relatively easy to obtain Pick's Theorem. Begin by observing that the vertices of the graph of P are the lattice points in the interior and on the sides of P, and the edges of the graph P are the sides of P and of the primitive triangles in the dissection of P. The f regions of the graph of P are the $f-1$ primitive triangles and the complement of P in the plane. Since each primitive triangle has area 1/2, we have

   $$A(P) = \frac{1}{2}f - 1.$$

   Let e_i denote the number of interior edges of primitive triangles, and let e_s denote the number of edges of primitive triangles on the sides of P. Explain why

   $$3(f-1) = 2e_i + e_s = 2e - e_s.$$

   Use Euler's Formula to rewrite this equation as

   $$f - 1 = 2(v-2) - e_s + 2.$$

   Finally, use

   $$A(P) = \frac{1}{2}(f-1)$$

   and

   $$v = B(P) + I(P)$$

   to obtain Pick's Theorem.