Pythagorean Points II: Sums of Squares

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There's a famous theorem (which we're not going to prove) that every positive integer is the sum of four squares. For example, you can write $7 = 2^2 + 1^2 + 1^2 + 1^2$; $25 = 5^2 + 0^2 + 0^2 + 0^2 = 4^2 + 3^2 + 0^2 + 0^2 = 4^2 + 2^2 + 2^2 + 1^2$; and so on. We're going to explore how many different ways a given number can be written as the sum of four squares. And keep that last question in mind: how will you count depending on whether the order matters (so the above representation for 7 counts as 4 ways) or doesn't matter (so there's just one way to write 7 as the sum of four squares).

1. Make a table showing how many ways there are to write each number as the sum of 1 square. (Yes, a lot of numbers can be written in zero ways!) (Are there already multiple interpretations for this problem?

2. In the next row of your table, show how many ways there are to write each number as the sum of 2 squares. Is there a pattern for which numbers can and cannot be written as the sum of 2 squares?

3. Now try 3 squares.

4. Finally, finish your table with 4 squares. You can see there are no more zeros in your table; can you prove there will never be?

5. What kinds of different answers can we uncover here?

6. Can we find a strange relationship with the sum of the divisors of the number?

Now we'll turn our attention to triangular numbers: 0, 1, 3, 6, 10, 15, 21, …

7. Make a table showing the number of ways to write each number as the sum of four triangular numbers. As a check, your two answers for 7 should be: if order doesn't matter, 2, namely 6+1+0+0 and 3+3+1+0, and if order does matter, 24.

8. Now it appears that three triangular numbers will suffice: that is, there are lots of numbers like 5 that cannot be written as the sum of two triangular numbers, but it looks like every number can be written as the sum of three triangular numbers. Can you prove it?

9. Double your number and add 1. Can you find a relationship to the number of ways of writing your number as a sum of triangular numbers?