Exploding Dots and Alien Arithmetic

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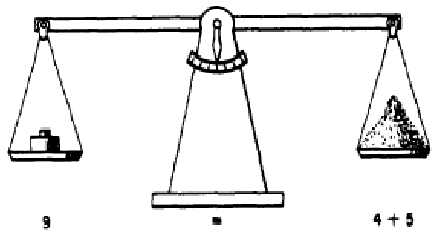
What is the smallest number of weights you need, so that you can weigh any integer number of grams of chocolate from 1 to 100 on a balance scale? Weights may be placed only on the left pan, and chocolate only on the right.



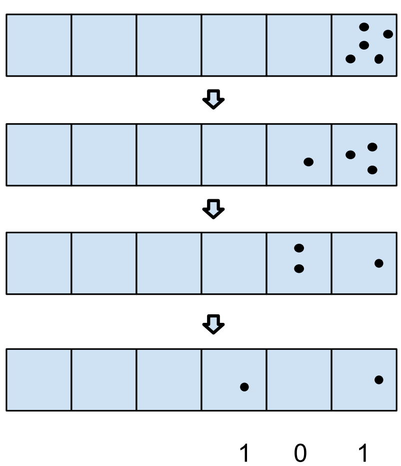
What if you are allowed to place the weights on either pan?

1 James Tanton's Exploding Dots

A $1 \leftarrow 2$ machine consists of a row of boxes, extending to the left as far as you'd like. To operate the machine, place a number of dots in the right most box. The machine then redistributes the dots according to the rule:

Two dots in any one box vanish (they explode) and are replaced with one dot one box to their left.

When all the explosions have died down, you can read off a code of 1's and 0's representing the number of dots in each box.



1. What happens if you start with six dots? Seven dots? Eight dots? 25 dots?
2. Does the order in which you do the explosions affect the final code?

3. What if you use a $1 \leftarrow 3$ machine instead? Try finding the codes for 13, 14, 15, and 20.

4. Write all the numbers from 19, 42, and 100, encoded using a $1 \leftarrow 6$ machine.

5. If your friend came up with the code 152 using a $1 \leftarrow 6$ machine, how many dots did she start with?

6. What is the code for 253 using a $1 \leftarrow 10$ machine?

2 Number Bases

From now on, a number written with a subscript will mean that that number is written using a the subscript as a base. That is, 11_6 means 11 base 6 – that is, the number encoded as 11 in a $1 \leftarrow 6$ machine, which is the number 7. A number without a base means ordinary base 10. So, for example, $20_6 = 12$ and $113_6 = 45$.

7. In ordinary base 10, we use string together the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in different ways to represent all the numbers. How many symbols do we need for a binary (base 2) number system? A base n number system?

8. Write these numbers in base 10:
   (a) 15_7
   (b) 35_7
   (c) 45_7
   (d) 412_7

9. Write these numbers in base 7:
   (a) 13
   (b) 48
   (c) 63
   (d) 625
   (e) 1000

10. Write in decimal (base 10) notation the numbers 10101_2, 10101_3, 211_4, 126_8.

11. Write the number 100_{10} in base 2, base 3, base 4, base 5, base 6, base 7, base 8, base 9.

3 Alien Arithmetic

12. Add $11121_3 + 120110_3$ (in base 3).

13. Multiply 102_3 by 201_3 (in base 3).

14. Write down the addition and multiplication tables in bases 2, 3, 4, 5.

15. Calculate

(a) $1100_2 + 1101_2$
(b) $1011_2 - 101_2$
(c) $100011_2 - 10100_2$
(d) $1011_2 \times 101_2$
(e) $1000_2 \div 11_2$

16. Calculate

(a) $101102_3 + 22012_3$
(b) $10120_3 - 212_3$
(c) $2012_3 \times 112_3$

4 Bigger Bases

17. Is it possible to write numbers in a base greater than 10, like 12? Try writing 2012_{10} in base 12.

18. Count to a hundred in base 12.

19. Write down the multiplication table in base 12.

20. Calculate

(a) $248_{12} + 9A7_{12}$
(b) $42_{12} \times 55_{12}$

5 Divisibility Rules

21. In base 10, you can tell if a number is even based on whether or not its last digit is even. State and prove a condition (involving the representation of a number) that allows you to determine whether a number is odd or even

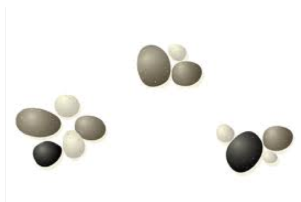
(a) in the base 3 number system
(b) in the base n number system

22. Find and prove a divisibility rule in base 7 arithmetic that is analogous to the rule (in ordinary base 10 arithmetic) for divisibility by 9. See if you can find other divisibility rules in base 7 arithmetic that are similar to rules for base 10.

6 The Game of Nim

23. One version of the game of Nim is played as follows by two players who alternate turns. There are two piles of stones, one pile with 10 stones and another one with 7. On each turn, a player can take any number of stones from either one of the two piles. The player who takes the last stone wins. Do you want to go first or second? What is the best strategy?

24. Another version of Nim is also played by two players, but this time with three piles of stones to start: one pile with 5 stones, a second pile with 4 stones, and a third pile with 3 stones. On each turn, a player can take any number of stones from any one of the three piles. The player who takes the last stone wins. What is the best strategy?



7 Just for Fun

25. You are a xeno-archeologist who has found an elementary school textbook from an ancient alien civilization. Although most of the book is no longer legible, you have found one equation that says: $3 \times 4 = 10$. How many fingers do you think the aliens have on each hand?

26. Does there exist a number system where the following equations are true simultaneously?
    (a) $3 + 4 = 10$ and $3 \times 4 = 15$?
    (b) $2 + 3 = 5$ and $2 \times 3 = 11$?

27. A blackboard bears a half-erased calculation exercise:

	2	3	?	5	?
+	1	?	6	4	2
	4	2	4	2	3

    What number system was used and what are the missing digits?

28. A spaceship full of hostile aliens is about to land on Earth. The aliens are very fond of Earth donuts, and you have persuaded them to leave you in peace in exchange for one donut for each alien on board. The captain radios down and says: "There are 100 of us total on board, and we would like 24 jelly donuts and 32 chocolate donuts with sprinkles." What number system is he using?

29. Going back to the exploding dots machines, experiment with a $2 \leftarrow 3$ machine. Encode the numbers 1 through 10 using this machine. What is the number base you are using?

Many of these problems are from *Mathematics Circles: the Russian Experience* by Fromkin, Genkin, and Itenberg