Graph Theory
2012.04.18

Our goal today is to learn some basic concepts in graph theory and explore fun problems using graph theory. A **graph** is a mathematical object that captures the notion of connection. It is a set of vertices (nodes) and edges (lines connecting two vertices), $G = (V, E)$.

DEFINITIONS

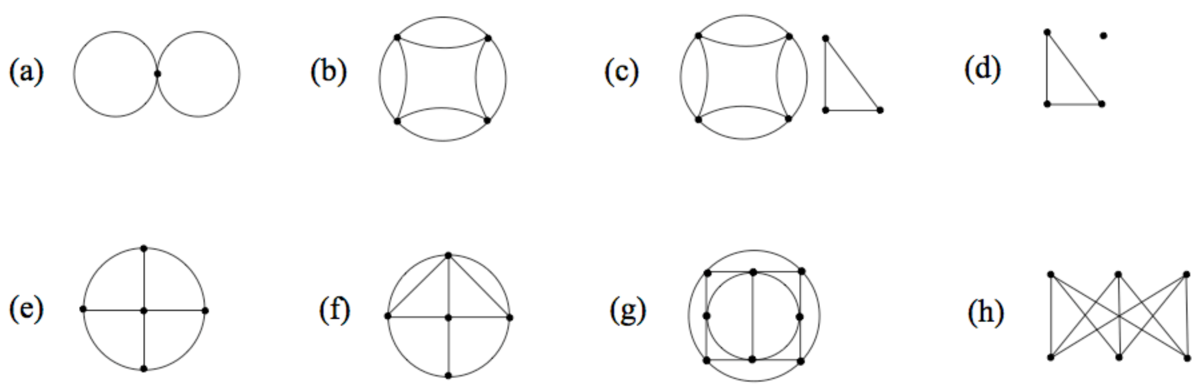
Finite graphs are graphs that have finite sets of vertices and edges.
When two endpoints of an edge are the same vertex, this is called a **loop**.
A graph is said to have **multiple edges** when more than one edge share the same set of endpoints.
The **degree** of a vertex is the number of edges coming out of that vertex.
A **simple graph** is a graph having no loops or multiple edges.
A **complete graph** is a graph where every pair of vertices is connected by a unique edge.
A **path** is an ordered "walk" along the graph starting at a vertex.
A **cycle** is a simple graph whose vertices can be cyclically ordered and each vertex has degree two.
A **connected graph** is a graph where there exist a path between any two vertices.
A **trail** is a path with no repeated edges.
A closed trail or a **circuit** is a graph where first and last vertices of a trail are the same.
An **Eulerian trail** is a trail in the graph which contains all of the edges of the graph.
An **Eulerian circuit** is a circuit in the graph which contains all of the edges of the graph.
A graph is **Eulerian** if it has an Eulerian circuit.
An **independent set** I is a set of vertices where no two vertices make up an edge.
A **matching set** M is a set of edges where no two edges share a common vertex.
A graph has **complete matching** if and only if every vertex of the graph touches exactly one edge of the matching.

EXERCISES

1. What is the difference between a path and a trail? Give examples of the two showing their difference.

2. What is the difference between a circuit and a cycle? Give examples of the two showing their difference.

3. Consider the following collection of graphs:



Which graphs are simple?

Suppose that for any graph, we decide to add a loop to one of the vertices. Does this affect whether or not the graph is Eulerian?

Which graphs are connected?

Which graphs are Eulerian? Trace out an Eulerian circuit or explain why an Eulerian circuit is not possible.

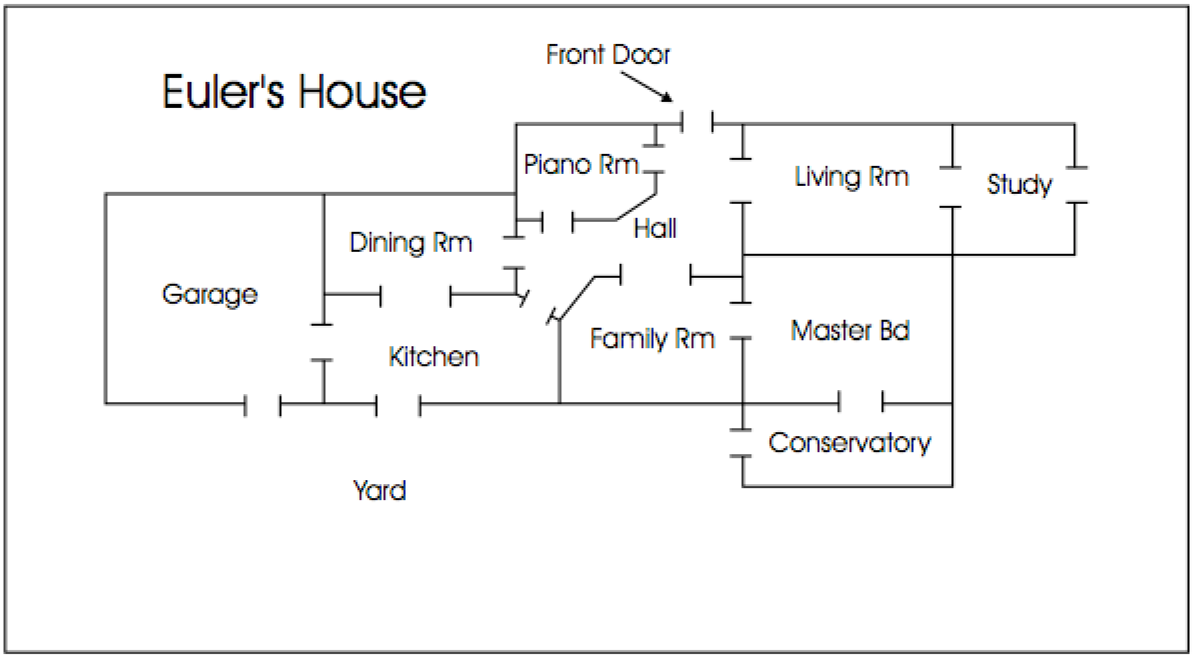
Are there any graphs above that are not Eulerian, but have an Eulerian trail? If so, which ones?

Give necessary conditions for a graph to be Eulerian.

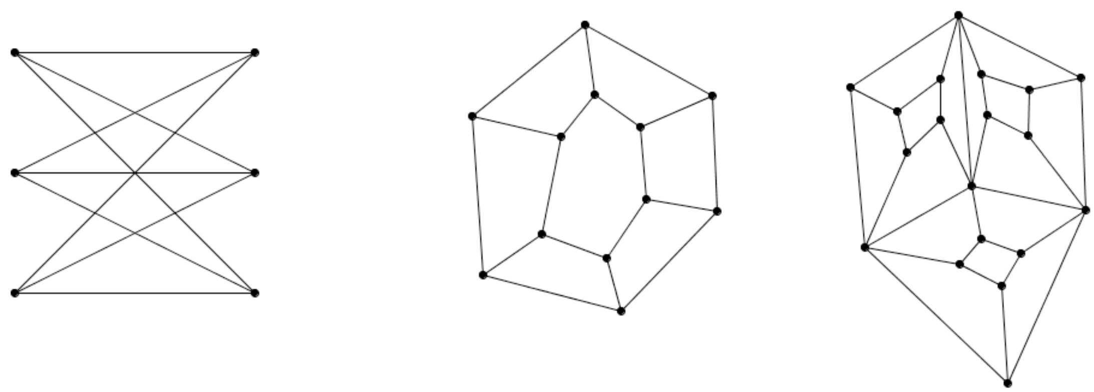
Give necessary conditions for a graph to have an Eulerian trail.

4. Given that a graph has an Eulerian circuit beginning and ending at a vertex, is it possible to construct an Eulerian circuit beginning and ending at any vertex in the graph? If so, give an example.

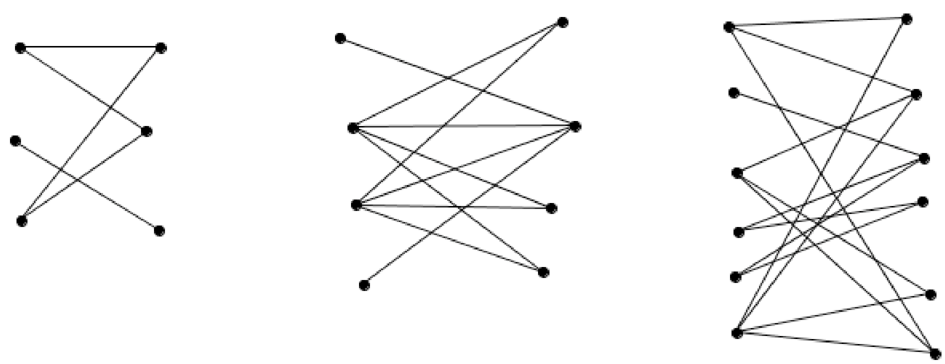
5. Euler's House. Baby Euler has just learned to walk. He is curious to know if he can walk through every doorway in his house exactly once, and return to the room he started in. Will baby Euler succeed? Redraw Euler's house as a graph. Can baby Euler walk through every door exactly once and return to a different place than where he started? What if the front door is closed?



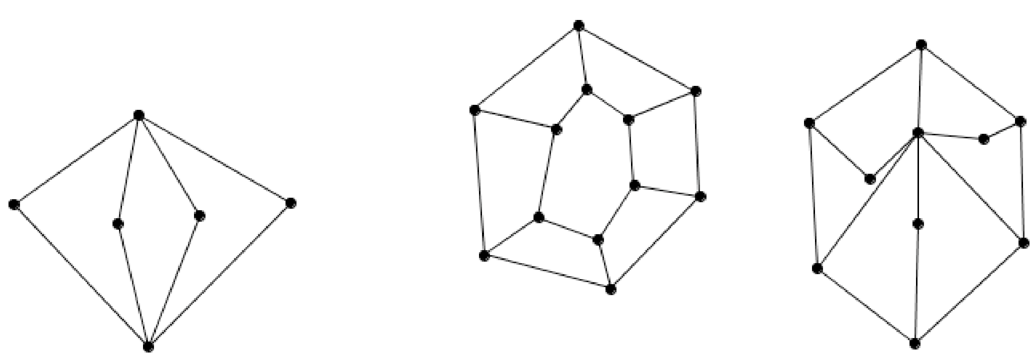
6. Suppose that there are three groups of tennis players as shown on the figure below. Each point represents each player. If two players know each other, there is a line that connects them. In each group, is it possible for players who know each other to form a team for tennis doubles? We say that, "to have a complete matching", no point must be left unmatched. Trace a complete matching for each graph below.



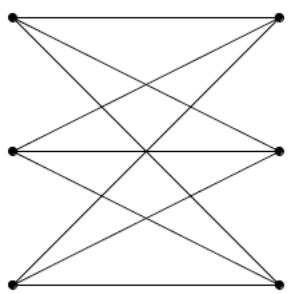
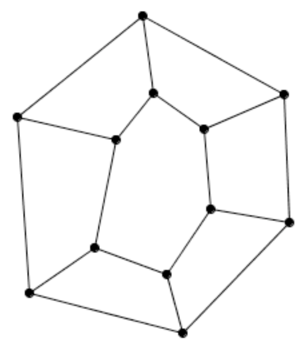
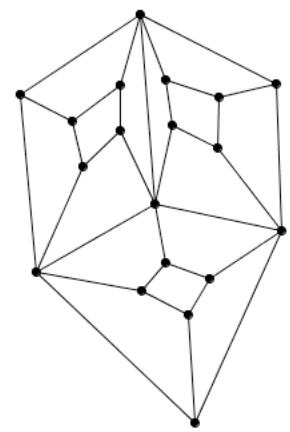
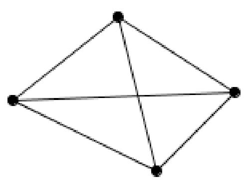
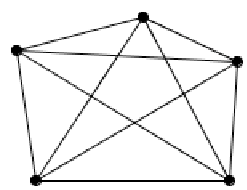
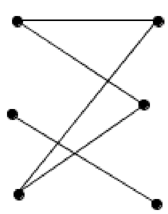
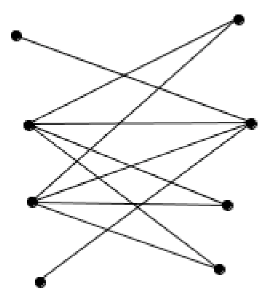
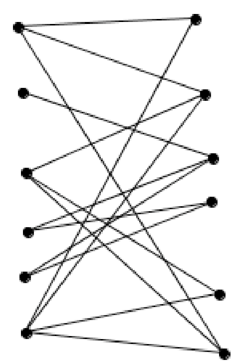
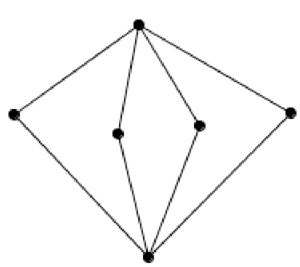
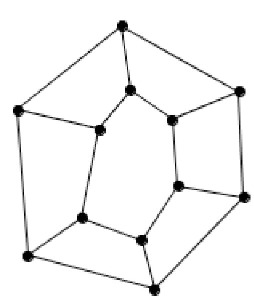
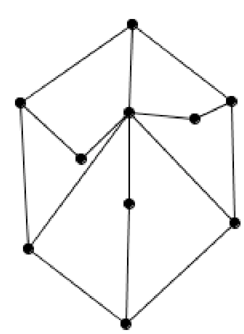
7. Do the groups in the figure below have a complete matching?



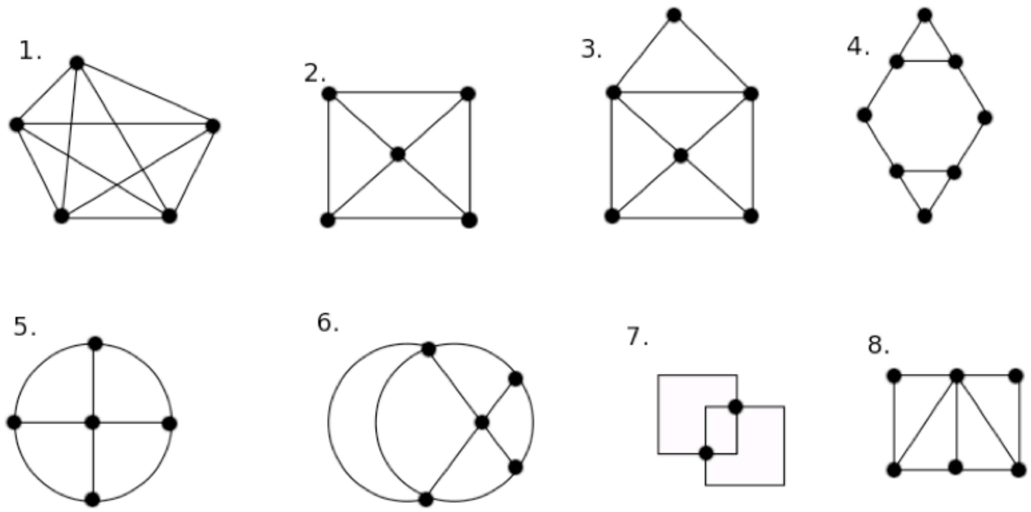
8. How about these figures? Do they have complete matching?



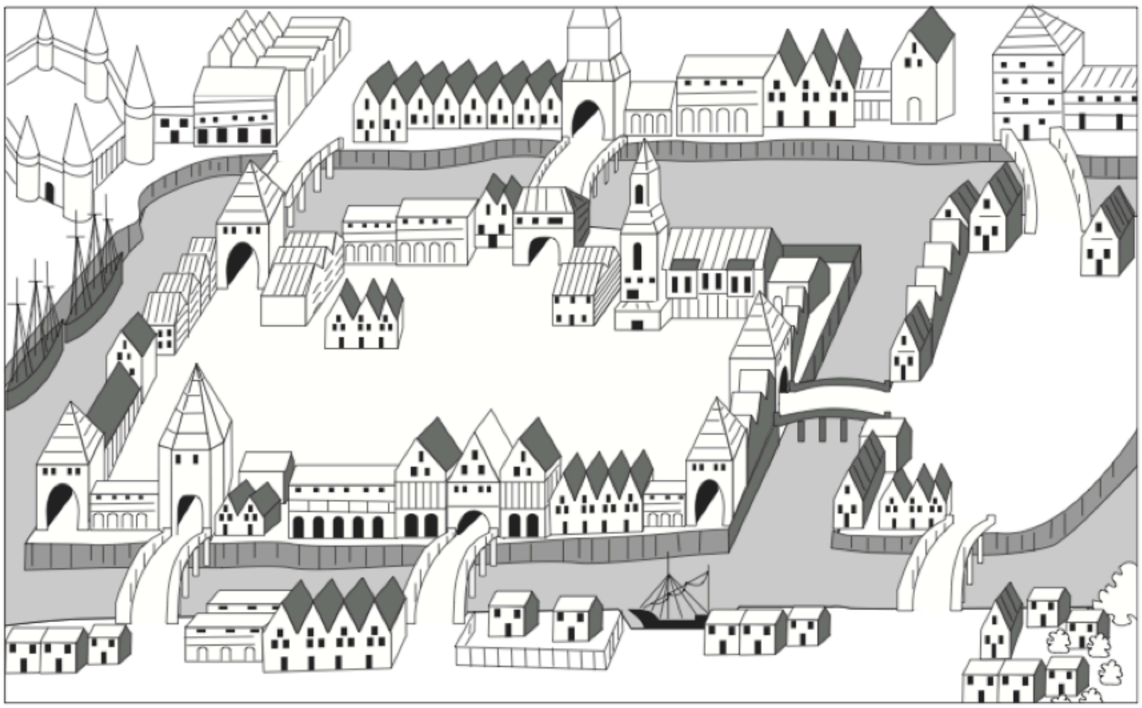
9. Using the same nine graphs and additional two graphs shown below, give one example of an independent set for each. If none exists, explain why.

10. For each graph below, determine (by tracing) whether is has an Eulerian trail, an Eulerian circuit, or both.



11. Bridges of Konigsberg.
Try to find a path through the city of Konigsberg that crosses each bridge only once (an Eulerian trail). Note that the river cuts the city in half, so one cannot travel "outside" of the picture to get from the bottom half to the top half. Represent this problem in a graph. Is it possible? Explain why or why not. If not possible, suggest an edge addition that would make it possible.



12. More Eulerian Trails. Determine whether each graph below has an Eulerian trail. If it does, trace it on the graph.

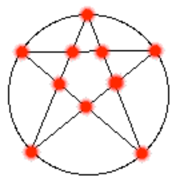


figure 1

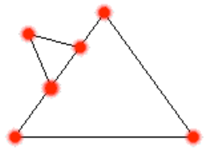


figure 2

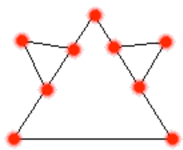


figure 3

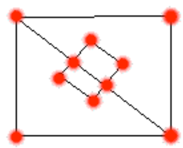


figure 4

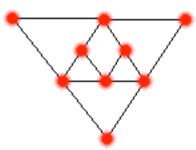


figure 5

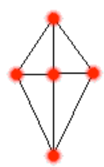
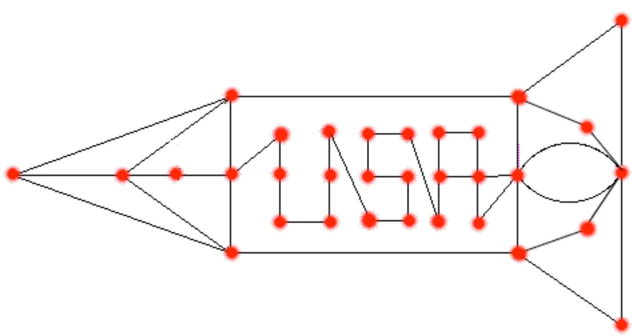
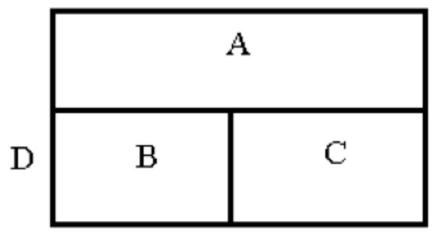
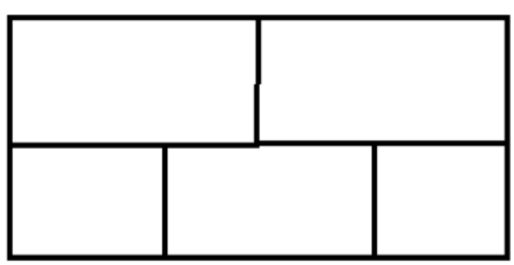


figure 6

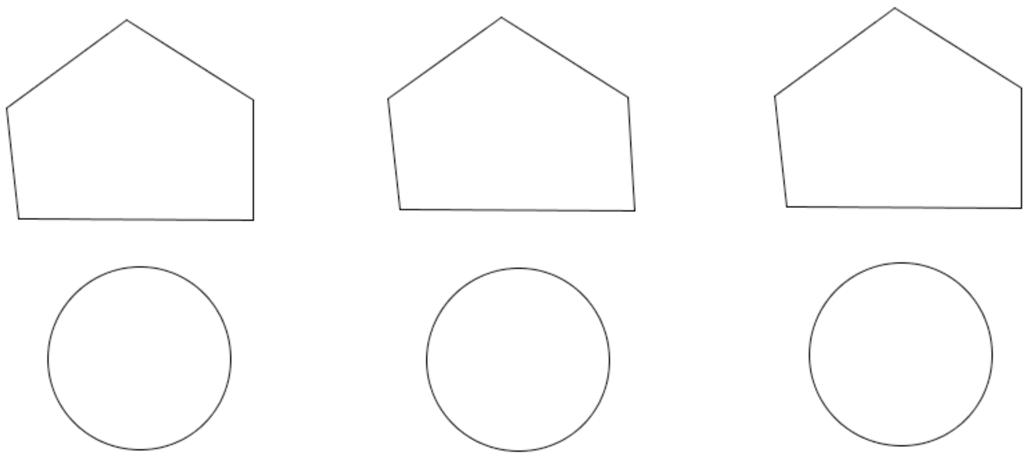


13. Draw an Eulerian trail that passes through each of the ten edges (line segments) of the following figures. First, represent these two figures as graphs then trace their Eulerian trails.





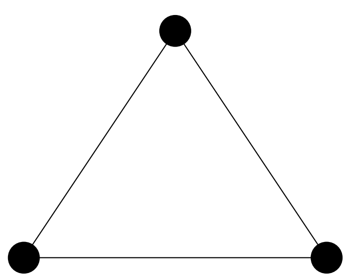
14. Connect three utilities (gas, water, electricity) to three houses without having any of the “wires” cross. Represent the picture below as a graph. Is it possible to connect them without crossing? Why or why not?



15. Suppose we have four colors: red, blue, green, yellow and we denote these colors by numbers such that

1 = red
2 = blue
3 = green
4 = yellow.

How many ways can you color the complete graph below if no two adjacent vertices can be colored with the same color? Now, suppose we represent the number of colors by the variable x. Can you come up with a polynomial that counts the number of ways to color the same graph?



16. A signed graph is a regular graph we've seen before, except now the edges are either a positive or a negative. Let us represent the vertices to be people, a negative edge to be a hostile relationship between two people and a positive edge to be a friendly relationship between two people. Draw a signed graph to show the famous quote "an enemy of my enemy is my friend".

17. A path graph P is a graph that starts at a vertex and takes a "path" and end at a terminal vertex where when we trace this with a pencil, we don't lift our pencil. For example, the graph below is a path graph from vertex a to vertex d. Show at least 5 paths of G other than the one shown.

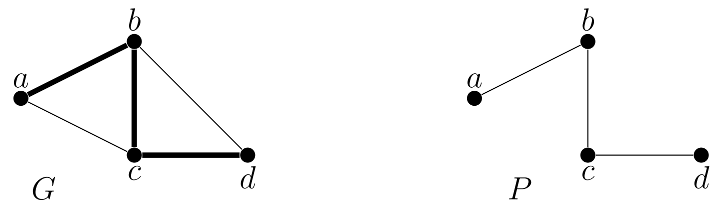


Figure 1: A path $P = abcd$ from vertex a to vertex d in G.

18. (50 pts.)Suppose we have the graph below. If we represent the X and Y vertices as axes on the xy-plane and denote available colors by numbers (see number 1), draw the xy-plane representation to show the number of ways to color the graph below with the same condition that adjacent vertices cannot be colored with the same color.

19. The Wolf, Cabbage, Goat and Farmer Problem
A farmer is bringing a wolf, a cabbage and a goat to market. The farmer arrives with all three at one side of a river that they need to cross. The farmer has a boat which can accommodate only one of the three (it's a big cabbage.) If the wolf is left along with the goat, the wolf will eat the goat. And, if the goat is left alone with the cabbage, the goat will eat the cabbage. The wolf can be left alone with the cabbage because wolves don't like cabbages. How can all three get across the river intact? Represent this as a graph.

20. Give two applications in the real world where graphs can be useful.