

Some of these games are from the book *Mathematical Circles: Russian Experience* by D. Fomin, S. Genkin, and I. Itenberg. Thanks to Paul Zeitz for the Puppies and Kittens game.

1 Penny Games

For each of these games, two players take turns moving. The winner is the last player who makes a legal move. See if you can find a winning strategy for one of the players. Try to prove that your strategy works. And always, try to generalize!

To describe a winning strategy, you have to explain what the winning player should do in order to win, no matter what the opponent does!

1. **One, Two, Three, Four, Takeaway**

- (a) There are 25 pennies on a table. On each turn, a player can take any number of pennies between 1 and 4. A player loses if he or she is unable to move (because there are no pennies left).
- (b) Same game as above but it starts with 24 pennies.
- (c) Same game again, only the initial number of pennies is n .

2. **Lame Tower.** On the top right square of an 8 by 8 chessboard there is a penny that can move either to the left or down through any number of squares. Players take turns moving the penny. A player loses if he or she is unable to move the penny (because it is already on the bottom left square). Consider various other initial positions of the penny.

3. **One, Two, Four Takeaway** There are 25 pennies in a pile. A player can take 1, 2, or 4 pennies on each turn. A player loses if he or she cannot continue (no more pennies left).

4. **Two Pile Nim.**

- (a) Now there are two piles of pennies, one pile with 10 pennies and another one with 7. On each turn, a player can take any number of pennies from either one of the two piles. The player unable to move (no pennies left) loses.
- (b) What about if the numbers of pennies in the piles are m and n ?

5. **Balance.** Players take turns putting pennies on a round table. The pennies cannot overlap and they cannot extend over the edge of the table. The player who cannot place another penny loses.

6. **Either, Or, Both.** There are two piles of pennies; one pile contains 10 pennies while the other contains 7. A player can take one penny from the first pile, or one penny from the second pile, or one penny from each of the two piles. The player unable to move loses.
7. **Heads and Tails** Two players take turns placing pennies on a 5×5 checkerboard. The first player puts the pennies down with the heads facing up, and the second player puts tails facing up. At the end of the play, the first player gets a point for each row or column that contains more heads than tails. The second player gets a point for each row or column that contains more tails than heads. The player with the most points wins.
8. **Puppies and Kittens.** There are two piles of pennies; one pile contains 10 and one contains 7. A player can take any number of pennies from the first pile, or any number from the second pile, or the player can take the same number of pennies from both piles. For example, a player could take 2 from the first pile, or 6 from the second pile, or 3 from each pile. The player unable to move loses.
9. **Three Pile Nim.** There are three piles of pennies; one pile with 6 pennies, a second pile with 5 pennies, and a third pile with 3 pennies. Two players take turns removing any number of pennies from any one of the three piles. The player unable to move loses.

2 Chocolate Bar Games

10. **Break the Bar.** You have a rectangular chocolate bar that is 6×8 squares in size. At each step, a player takes one piece of the chocolate and breaks it in two along a single straight line bounded by the squares. For example, you can turn the original bar into a 6×2 piece and a 6×6 piece, and this latter piece can be turned into a 1×6 piece and a 5×6 piece. The player who cannot make any more breaks loses.
11. **Free a Square.** Two players take turns breaking a piece of chocolate consisting of 5×10 small squares. At each turn, they may break along the division lines of the squares. The player who first obtains a single square of chocolate wins.

3 Number Games

12. **Build a Number.**

- (a) Two players want to build a 7-digit number, writing one digit at a time from left to right. The first player wins if the number is divisible by 9; the second player wins if the number is not divisible by 9.
- (b) What happens if 9 is replaced by 11 in the previous game?

13. **Pluses and Minuses.**

- (a) There are 12 minuses written along a line. A player replaces either one minus by a plus or two adjacent minuses by two pluses. The player who replaces the last minus wins.
- (b) Same game as above, only the minuses are written around a circle.

14. **Subtract a Divisor.** At the start of the game, there is a number 60 written on the board. On each turn, a player can reduce the number that is currently on the board by any of its positive divisors. If the resulting number is 0, the player loses.

15. **Addition to 1000.** The game begins with the number 2. In one turn, a player can add to the current number any natural number smaller than it. The player who reaches the number 1000 wins.

4 Nim and Jim

16. **Three Pile Nim.** There are three piles of pennies; one pile with 6 pennies, a second pile with 5 pennies, and a third pile with 3 pennies. Two players take turns removing any number of stones from any one of the three piles. The player unable to move loses.
17. **Three Row Jim.** Jim was invented by Bard math professor Japheth Wood. A Jim game starts with three rows of X's and O's. Players alternate moves: select a row, and change one or more symbols on that row (X's to O's or O's to X's). RULE: The first change from the left must be X to O (but it does not need to be the leftmost X). The last player with a legal move wins. Equivalently, if you only O's, then you've lost!

For example:

X X O		X O O	
O X O	→	O X O	This is a legal move.
O O X		O O X	
X X O		X O X	
O X O	→	O X O	This is a legal move.
O O X		O O X	
X X O		X X O	
O X O	→	X X O	This is NOT a legal move.
O O X		O O X	

5 Birthday Trick

1	3	5	7
9	11	13	15
17	19	21	23
25	27	29	31

2	3	6	7
10	11	14	15
18	19	22	23
26	27	30	31

4	5	6	7
12	13	14	15
20	21	22	23
28	29	30	31

8	9	10	11
12	13	14	15
24	25	26	27
28	29	30	31

16	17	18	19
20	21	22	23
24	25	26	27
28	29	30	31

I will guess the day of the month of your birthday!

6 A Game Tournament

1. **Three Pile Nim.** There are three piles of pennies; one pile with 8 pennies, a second pile with 7 pennies, and a third pile with 6 pennies. Two players take turns removing any number of pennies from any one of the three piles. The player unable to move loses.
2. **Three Pile Nim.** Same game, but start with piles of size 13, 11, and 6.
3. **Pluses and Minuses in a Line.** There are 10 minuses written along a line. A player replaces either one minus by a plus or two adjacent minuses by two pluses. The player who replaces the last minus wins.
4. **Pluses and Minuses in a Circle.** Same game as above, only the minuses are written around a circle.
5. **Game of 23.** This game begins with the number 0. In one turn, a player can add 1, 2, 3, or 4 to the current number. The player who reaches 23 wins.
6. **Divisible by 9.** Two players want to build a 7-digit number, and they take turns writing one digit at a time from left to right. The first player wins if the number is divisible by 9; the second player wins if the number is not divisible by 9.
7. **Divisible by 11.** What happens if 9 is replaced by 11 in the previous game?
8. **Subtract a Divisor.** At the start of the game, there is a number 60 written on the board. On each turn, a player can take any positive divisor of the current number and subtract that divisor from the current number. For example, the first player can subtract 12 from 60 to get 48. The player who does a subtraction that results in the number 0 loses.
9. **Addition to 1000.** The game begins with the number 2. In one turn, a player can add to the current number any natural number smaller than it. The player who reaches the number 1000 wins.
10. **Either, Or, Both.** There are two piles of pennies; one pile contains 10 pennies while the other contains 6. A player can take one penny from the first pile, or one penny from the second pile, or one penny from each of the two piles. The player unable to move loses.
11. **Puppies and Kittens.** There are two piles of pennies; one pile contains 10 and one contains 6. A player can take any number of pennies from the first pile, or any number from the second pile, or the player can take the same number of pennies from both piles. For example, a player could take 2 from the first pile, or 6 from the second pile, or 3 from each pile. The player unable to move loses.

linda@marinmathcircle.org