The Pigeonhole Principle

Marin Math Circle

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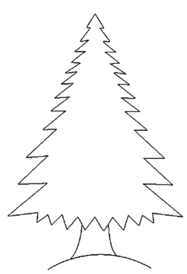
Many of these problems are from *Mathematical Circles (Russian Experience)* and from *A Decade of the Berkeley Math Circle -Volume 1*

1. I own 7 pairs of socks and each pair is a different color. If all 14 socks are loose in the dryer, how many will I have to pull out to guarantee that I get at least two of the same color?

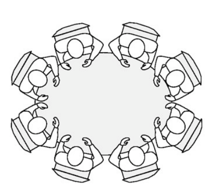
2. Over a million Christmas trees were sold in California this winter. No tree has more than 800,000 needles on it. Show that two of the Christmas trees had the same number of needles on them at midnight of the night before Christmas.



3. A bag contains 10 black marbles and and 10 white marbles. What is the smallest number of marbles that you must pull out to guarantee that you get at least two marbles of the same color?

4. The population of the Bay Area is about 7.4 million people. Show that at least two people in the Bay Area have the exact same number of hairs on their heads. Assume that no person has more than a million hairs on their heads. (The average number of hairs on a human head is about 100,000.)

5. Eight chairs are set around a circular table. On the table are name placards for eight guests. After the guests are seated, it is discovered that none of them are in front of their own names. Show that the table can be rotated so that at least two guests are simultaneously correctly seated.



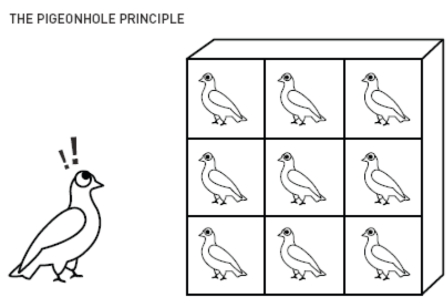
6. Given 12 integers, show that two of them can be chosen whose difference is divisible by 11.

7. Sixteen boxes of chocolate are for sale at the store. The chocolates are of three different kinds (dark, milk chocolate, and white chocolate), and all chocolates in a box are of the same kind. You want to buy 6 boxes of chocolates to give to your 6 cousins, but you want to give them all the same kind of chocolate so there won't be any squabbling. Is this necessarily possible?

Pigeonhole Principle:

a) If you put $n+1$ or more pigeons into n pigeon holes, at least one pigeon hole must contain more than one pigeon.

b) If you put $kn+1$ or more pigeons into n pigeon holes, at least one pigeon hole must contain more than k pigeons.

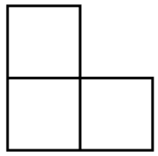


8. How many students do you need to have in a class to guarantee that at least 4 students will have birthdays in the same month?

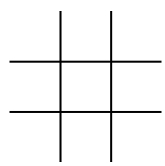
9. Show that in any group of five people, there are two who have an identical number of friends within the group. (Friendship is mutual – if A is B's friend, then B is A's friend.)

10. Several soccer teams enter a tournament in which each team plays every other team exactly once. Show that, at any moment during the tournament, there will be two teams which have played, up to that moment, the same number of games.

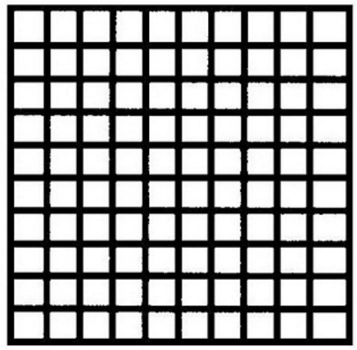
11. What is the largest number of squares on an 8×8 checkerboard which can be colored green, so that in any arrangement of three squares (a "tromino" as drawn below), at least one square is not colored green? (The tromino may be appear as in the figure or it may be rotated through some multiple of 90 degrees.)



12. What is the smallest number of squares which can be colored green, so that in any tromino at least one square is colored green?
13. What is the largest number of kings that can be placed on a chessboard so that no two of them are attacking each other?
14. Fifty-one points are scattered within a square of side length one meter. Show that at least 3 of the points can be covered with a square of side length 20 cm.
15. Show that an equliateral triangle cannot be covered completely by two smaller equilateral triangles.
16. Each box in a 3×3 tic-tac-toe board is filled with one of the numbers -1, 0, 1. Prove that of the eight possible sums along the rows, the columns, and the diagonals, two sums must be equal.



17. Of 40 children seated at a round table, more than half are girls. Prove that there are two girls who are seated diametrically opposite each other.
18. Given eight different positive integers, none greater than 15, show that at least three pairs of them have the same positive difference. (The pairs may overlap – that is, two pairs or all three pairs may contain the same integer.)
19. Prove that there exist two powers of two which differ by a multiple of 2013.
20. Prove that there exists an integer whose decimal representation consists entirely of 1's, and which is divisible by 2013.



10 x 10 GRID

21. Integers are placed in each square of a 10×10 chessboard, in such a way that no two neighboring integers differ by more than 5. (Two integers are considered neighbors if their squares share a common edge.) Prove that two of the integers must be equal.

22. Prove that among any six people, there are either three people who all know each other or three people who are all strangers to each other. (Assume that if person A knows person B, then person B also knows person A.)

23. Five lattice points are chosen on an infinite square lattice. Prove that the midpoint of one of the segments joining two of these points is also a lattice point.

24. Prove that of any 52 integers, two can always be found such that the difference of their squares is divisible by 100.

25. Prove that you can choose a subset of ten given integers such that their sum is divisible by 10.

26. Given 11 different positive integers, none greater than 20, prove that two of these can be chosen, one of which divides the other.

27. Eleven students have formed five study groups. Prove that two students can be found, say A and B, such that every study group which includes student A also includes student B.