Foreheads - consecutive numbers

Annie and Zoe have *consecutive* natural numbers written on their foreheads. This is common knowledge (they both know this, and they both know they know this, etc.) We ask them in turn if they know their own number: “Annie, do you know?” - “Zoe, do you know?” - “Annie, do you know now?” - etc. They only answer “Yes, I do know” or “No, I don’t”, and we keep asking until someone figures out their own number (it turns out that the other person then does, too, right on the next question).

1. If Annie has 2 and Zoe has 1, what happens?
   - Annie sees 1 so she knows she must have 2, so she answers “I do know”. **1 Question.**
2. If Annie has 1 and Zoe has 2, what happens?
   - Annie sees a 2 so she says “I don’t know”, realizing she could have 1 or 3. Zoe sees a 1 so she says “I do know”. **2 Questions.**
3. If Annie has 3 and Zoe has 2, what happens?
   - Annie sees 2 and says “I don’t know”, realizing she may have 1 or 3. Zoe sees 3 so she also says “I don’t know” - she could have 2 or 4 and in either case she’d expect Annie to say what she did. Now Annie realizes she can’t have a 1, so she must have 3, so she says “I do know now”. **3 Questions.**
4. If Annie has 2 and Zoe has 3, what happens?
   - Annie sees 3 and says “I don’t know”.
   - Zoe sees 2 and realizes that if she had 1, Annie would know. So she knows she must have 3. “I do know”.
   - **2 Questions.**
5. If Annie has 4 and Zoe has 3, what happens?
   - Annie sees 3. “I don’t know”.
   - Zoe sees 4. “I don’t know”.
   - Annie realizes she doesn’t have 2 (otherwise we would be in the previous situation) so she knows she has 4 and says “I do know”.
   - **3 Questions.**
6. If Annie has 100 and Zoe has 101, what happens?
   - Using the small cases we’ve solved, we can formulate a conjecture about the number of questions required, and then prove it.
   - Whoever sees the smaller number is always the first to figure out their own number. Thus if Annie has the smaller number, the number of questions is always even, and if Zoe has the smaller one, the number of questions is odd. In fact, the number of questions is always one of the numbers, even or odd according to that rule.
   - So for (100, 101) the answer is 100 questions. It’s astonishing that both Annie and Zoe know right away that this is going to take roughly 100 questions, yet they have no choice but to go through the motions of saying “No.” - “No.” -”No.” etc. They *know* this is going to happen, yet they have no way of “skipping ahead”.

The General Case (Spoilers!)

Let f(a,b) be the number of questions required if Annie has a, Zoe has b and Annie is the first person to get asked. We can prove by induction that:

f(2k, 2k+1) = 2k
f(2k+1, 2k) = 2k+1
f(2k+2, 2k+1) = 2k+1
f(2k+1, 2k+2) = 2k+2

In other words,

f(n, n+1) = n or n+1, whichever is even
f(n+1, n) = n or n+1, whichever is odd

1 2: 2	2 1: 1
2 3: 2	3 2: 3
3 4: 4	4 3: 3
4 5: 4	5 4: 5
5 6: 6	6 5: 5
6 7: 6	7 6: 7

Red and White Hats

The girls in the room have red or white hats. They can see each other's but not their own. Every whole hour on the hour, whoever knows her own color stands up and says "Got it!" (if a girl figures out her own hat color as a result of some other girl standing up, she must keep a poker face and wait for the next round to announce it).

1. Two girls, one red hat, one white hat. Does anything ever happen? (no). A person walks in the room at 12:30 and says "Oh, nice to see one-or-more red hats". What happens?
2. Two girls, both have red hats. A person walks in and makes the same statement. What happens? How can that be, given that both girls knew about a red hat long before the statement was made?
    - This is the crucial step. Both girls know that there's a red hat in the room, but neither of them knows if the other knows that. They both

know it, but it's not common knowledge. The statement made by the person changes that.

  - Before explaining this point of view, it's fun to challenge the kids to explain how this can be. The discussion turns rather lively. One challenge is that seeing how counterintuitive this is, some kids are led to believe that the girls will figure out the hat colors even **before** any statement is made (on the grounds that, since what the person said was known, they could have *pretended* they said it anyway). This is not true but it's somewhat tricky to convince someone otherwise, and proving it formally is harder than it seems.

3. Three girls, all have red hats. What happens?
4. Ten girls, 3 have red hats, 7 have white hats. What happens?
5. Check out Terry Tao's blog post here. You can now determine which of the solutions is the correct one, and why.
6. Almost all discussions of this puzzle focus on upper-bound proofs, which demonstrate that a certain sequence of logical deductions lead the participants to infer their own hat color within a certain number of rounds. But there's also an implicit lower-bound question: How do we know that no participant can infer their own hat color in *fewer* rounds? To prove this we need to rule out all possible logical deductions, for which we need to build models which show that there are consistent possible states of the world at all times prior to the upper-bound time in which the hat colors are indeterminate. This is in fact much harder, and is discussed by Terry Tao in a separate blog post here.

Foreheads - the sum game

Alice, Bob and Charlie have natural numbers x,y,z on their foreheads with x+y=z. We ask them in turn if they know their own number.

Alice=1, Bob=1, Charlie=2: what happens? How about 10, 10, 20? Generally, what happens if two numbers are the same?

Alice=1, Bob=2, Charlie=3: what happens? How about 10, 20, 30? Generally, what happens if one number is double another?

Alice: "No". Bob: "No". Charlie: "No". Alice: "No". Bob: "No". Charlie: "Yes, it's 12". What are the numbers?

- We're trying to determine the set of numbers on Alice's and Bob's foreheads. We don't expect to determine which of them has which number.
- This is a somewhat harder problem. Think about what you've learned, and what ratio between numbers would yield two rounds of "No" for Bob.

Sum and Product Puzzles

Someone picks two integers, X and Y, each from the interval 2 to 99, and tells Mr. Sam the sum of the two integers, and Mr. Paul the product of the two. Sam and Paul do not know the values given to the other. Paul tells Sam, “I do not know the two integers.” Sam tells Paul, “I knew you wouldn’t. Neither do I.” Paul replies, “Oh, now I know the integers.” Sam replies back, “Now I know too.”

Given that both of them are telling the truth, what are the two integers?

This famous puzzle appears in many places. Numerous variants can be found here. *Warning*: the first version, by Martin Gardner, is wrong. A nice discussion of this mistake and its resolution by Lee Sallows is here.

Some hints to get you started:

- If X and Y are both primes, then P=XY is uniquely factorable so Paul would know the numbers right away.
- Consequently, if S=X+Y can be written as the sum of two primes, Sam could suspect that P is uniquely factorable and would not be able to know that Paul can’t know the numbers. So S cannot be the sum of two primes; in particular it cannot be an even number (why?), nor a prime+2.

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Here’s another variant, even harder:

P and S are given the product and sum of two non-zero digits (1 to 9).
1. P says "I don't know the numbers".
   S says "I don't know the numbers".
2. P says "I don't know the numbers".
   S says "I don't know the numbers".
3. P says "I don't know the numbers".
   S says "I don't know the numbers".
4. P says "I don't know the numbers".
   S says "I don't know the numbers".
5. P says "I know the numbers".

This one is from mathpuzzle. The solution is 2 and 8.