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TO INFINITY AND BEYOND

1. Achilles and the Turtle (Zeno's paradox) Achilles runs 1 mile a minute, and he is 1 mile behind the Turtle, which is 100 time slower than him. Will he ever catch up with the Turtle?

Zeno's answer: No. As A covers the mile, T is .01 miles ahead, when A covers this .01 miles, T is .0001 miles ahead, and so on, so T is always ahead of A.

Well, on the other hand, in 2 minutes, A will be .98 miles ahead of T. Contradiction!

How to resolve it? *Hint:* 1.0101010101.... = 100/99.

2. The golden ratio and continued fractions:

$$\phi := \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}} = ?$$

Let's try: $1/1 = 1$, $1/(1+1) = 1/2$, $1/(1+1/(1+1)) = 2/3$,
$1/(1+1/(1+1/(1+1))) = 3/5$
$1/(1+1/(1+1/(1+1/(1+1)))) = 1/(1+3/5) = 5/8$, Do you recognize the pattern?

On the other hand, $\phi = 1/(1+\phi)$, or $\phi^2 + \phi - 1 = 0$, or

$$\phi = \frac{-1 \pm \sqrt{1+4}}{2}, \quad \text{taht is} \quad \phi = \frac{\sqrt{5}-1}{2}.$$

3. The geometric series. $S := 1 + q + q^2 + q^3 + \cdots + q^n + \cdots = ?$.
We have: $S = 1 + qS$, or $(1-q)S = 1$, or $S = 1/(1-q)$.

Example: $1.01010101\cdots = 1/(1-.01) = 1/.99 = 100/99$.

Exercise: $1 + q + q^2 + \cdots + q^n = (1-q^n)/(1-q)$.

4. "Infinite" numbers: $s = \ldots 11111 = ?$.

Let's try: $\ldots 11111 \times 9 + 1 = \ldots 00000$, and hence $s = -1/9$.

Another way: $1 + 10 + 10^2 + 10^3 + \cdots + 10^n + \cdots = 1/(1-10) = -1/9$.

Exercise: Show that infinite numbers can be added and multiplied using the standard algorithms.

Examples: $\ldots 423423423 \times 5 = ?$, $\ldots 11111 \times \ldots 11111 = ?$

5. Dyson numbers. Find $D = 1??...??x$ such that when the rightmost digit is moved to the leftmost place, the number doubles: $x1??..?? = 2D$.

$$\begin{array}{r} \dots 105263157894736842 \\ \times 2 \\ \dots 210526315789473684 \end{array}$$

Q: What is the infinite repeating number $D_\infty := \dots 105263157894736842$ equal to?

A: $2D_\infty = (D_\infty - 2)/10$, or $D_\infty = -2/19$.

6. Is $7 \times 7 = 47$? Find all k-digit numbers, such that if an integer x ends on the right with these k digits, then x^2 also ends with the same k-digits: $(\dots x_k \dots x_1)^2 = (\dots x_k \dots x_1$.

$k = 1$: $0^2 = 0$, $1^2 = 1$, $5^2 = 25$, $6^2 = 36$.

For the ending 6:

$k = 2 : (6 + 10? + ...)^2 = 36 + 120? + ... = 6 + 10? + ...)$ or $11? + 3 \equiv ? + 3 \equiv 0 \mod 10$ or $? = 7$.

$k = 3 : (76 + 100? + ...)^2 = 776 + 1200? = 76 + 100?$ or $? + 7 \equiv 0 \mod 10$ or $? = 3$.

$k = 3$: $376^2 = 141376$, hence $? + 1 \equiv 0 \mod 10$ or $? = 9$.

$k = 4$: $9376^2 = 87909376$, hence $? + 0 \equiv 0 \mod 10$ or $? = 0$.

$k = 5$: $09376^2 = ..909376$, hence $? + 9 \equiv 0 \mod 10$ or $? = 1$.

$k = 6$: $...109376^2 = ...3109376$, hence $? + 3 \equiv 0 \mod 10$ or $? = 7$.

$x = ...7109376$

For the ending 5:

$k = 2 : (5 + 10? + ...)^2 = (25 + 100? + ... = 5 + 10?$ hence $? = 2$.

$k = 3 : (25 + 100? + ...)^2 = 625 + 1000? + ... = 25 + 100?$ hence $? = 6$.

And so on: $625^2 = 390625$ hence the next digit is 0, and the next is 9, then $90625^2 = ...890625$ hence the next digit is 8.

$x = \dots 56259918212890625 = (((((5^2)^2)^2)^2)^2)^{\cdots}$

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7. Why does the quadratic equation $x^2 = x$ have 4 roots among infinite numbers?

Answer: higher mathematics (p-adic numbers): $\mathbb{Z}_{(10)} = \mathbb{Z}_{(2)} \times \mathbb{Z}_{(5)}$, and $x^2 = x$ has solutions $(0,0)$, $(1,1)$, $(1,0)$, and $(0,1)$, where $(1,0)$ is the number which has remainder 1 modulo $2, 2^2, 2^3, 2^4, ...$ and 0 modulo $5, 5^2, 5^3, 5^4, ...$, while $(0,1)$ the other way around: remainder 0 modulo $2, 2^2, 2^3, 2^4, ...$, and 1 modulo $5, 5^2, 5^3, 5^4, ...$